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Incorporation of internal state variables and plastically-induced anisotropy into a constitutive model involving scalar, conjugate, stress/strain base pairs

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ABSTRACT

In a previous study from the author and his co-worker, a constitutive model for isotropic, elastic perfectly-plastic materials was developed using scalar, conjugate, stress/strain base pairs. These stress/strain base pairs result from a Gram–Schmidt decomposition of the deformation gradient. A limitation of this prior work is that we assumed the microscopic structures of such elastic–plastic materials would remain constant throughout a deformation process, *i.e.*, there would be no change in the resulting microstructural properties. Typically, internal state variables are used to represent macroscopic manifestations of these microstructural properties. In this paper, we incorporate internal state variables into our previously developed constitutive model, and as a consequence, plastically-induced anisotropy shows up naturally in the developed model. These are, however, different from the typically used internal state variables and are introduced only for this purpose.

1. Introduction

In recent years, a novel **QR** decomposition of the deformation gradient has been proposed that has several advantages over the traditional polar decomposition. In this decomposition, the matrix of a deformation gradient is decomposed into an orthogonal rotation $\boldsymbol{\mathcal{R}}$ and an upper-triangular matrix $\boldsymbol{\mathcal{U}}$, called the Laplace stretch (Freed et al., 2019). Srinivasa (2012) showed that this decomposition is particularly useful from an experimental standpoint, as one can directly and unambiguously measure the components of Laplace stretch in a particular coordinate system, obtained through Laplace's technique of successive orthogonal projections. The **QR** kinematics have been further explored by Freed and Srinivasa (2015), Rajagopal and Srinivasa (2016), Freed and Zamani (2018), Paul and Freed (2020b), and Clayton (2020). Based on this kinematics, constitutive models have been developed using scalar, conjugate, stress/strain, base pairs for 2-D planar membranes (Freed, 2017), 3-D isotropic and anisotropic solids (Freed et al., 2017), and elastic solids that exhibit Kelvin–Poisson–Poynting effects (Freed and Zamani, 2019). The recent surge in the use of **QR** decomposition can be attributed to its various advantages over the traditional polar decomposition of the deformation gradient. To begin with, the **QR** kinematics offer a simple framework that is easy to implement both from a computational and an experimental standpoint (see detailed discussion in Srinivasa (2012)). Unlike the traditional theory, the **QR** framework does not require computationally expensive eigenvalue analysis to derive the invariants of the relevant kinematic quantities, used in different constitutive models (Mooney, 1940; Rivlin, 1948). The kinematic variable such as the stretch tensor $\boldsymbol{\mathcal{U}}$ is easy to visualize as opposed to its polar decomposition counterpart, $\mathbf{U}$. Moreover, the use of **QR** kinematics paved way to develop constitutive models using scalar conjugate stress/strain base pairs. The scalar nature of the stress and strain attributes significantly reduces the complexity in analysis as well as computational cost. In a series of papers, Criscione et al. (2000), Criscione (2004) have shown that the orthogonality of relevant kinematic quantities plays an important role in reducing the experimental error (see details in Section 2.2). The minimum covariance between the strain attributes in the **QR** framework has been shown to perform satisfactorily in this context (Freed, 2017); it is also relatively easy to implement as compared to the proposed alternative invariants of Criscione et al. (2001), Criscione and Hunter (2003). These advantages have been explored in recent papers (Kazerooni et al., 2019; Jiang et al., 2023) where a suitable experimental protocol, including parameter identification, has been established to validate the constitutive models based on a **QR** decomposition.

This kinematics was further extended to elastoplasticity by Freed et al. (2019) when they decomposed Laplace stretch into its elastic and plastic components, *i.e.*, $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{U}}^e\,\boldsymbol{\mathcal{U}}^p$, using the property that the set of all upper-triangular matrices forms a group under matrix multiplication. Traditionally a multiplicative decomposition of the deformation gradient $\mathbf{F} = \mathbf{F}^e\,\mathbf{F}^p$, first proposed by Bilby et al. (1957) and popularized by Kröner (1959), Lee and Liu (1967) and Lee (1969), is used to

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describe the plastic behavior of materials in a finite deformation setting. This decomposition assumes the existence of an intermediate, stress free configuration obtained by a tangent map $\mathbf{F}^p$ from the reference configuration of the body. Although this decomposition has been in use since the 1960s, different aspects of this theory have often led to contentions among researchers. In fact, Naghdi (1990) explained in his long review paper that the only areas of agreement are the need for — developing a suitable principle such as a flow rule or a strain hardening law, defining an appropriate measure for the permanent deformation or, a criterion to distinguish between the elastic and plastic regimes. The disagreements over fundamental aspects of the decomposition, starting from the existence of a stress free intermediate configuration (Green and Naghdi, 1971; Casey and Naghdi, 1980) to the definition of an appropriate plastic strain measure (Green and Naghdi, 1968; Nemat-Nasser, 1979), are quite commonplace and have been categorically documented by Naghdi (1990). Several attempts have been made to propose new frameworks that do not directly invoke the idea of a $\mathbf{F}^e\,\mathbf{F}^p$ decomposition but use alternative techniques to address some of these issues, *e.g.*, Naghdi and Trapp (1974), Miehe (1998).

Although the framework proposed by Freed et al. (2019) does not resolve to fix all these theoretical issues,[1] it is an important attempt in three key aspects– *(i)* the uniqueness of the elastic–plastic decomposition is ensured, *(ii)* it helps us understand the elastic–plastic decomposition in a simplified manner and hence, proves to be an useful tool for experiments and, *(iii)* it leads to a much simpler constitutive model based on *scalar* conjugate stress/strain base pairs that reduces complexity in analysis as well as computational cost. The traditional $\mathbf{F}^e\,\mathbf{F}^p$ decomposition assumes the existence of "a collection of local configurations" that are related to the undeformed and deformed configuration of the body through the tangent maps $\mathbf{F}^p$ and $\mathbf{F}^e$, respectively. These maps, in general, are not integrable and thus, are not gradients of any deformation maps. Naturally, the lack of associated deformation maps hinders one to understand (or visualize) the corresponding deformation processes. This is where Freed et al. (2019)'s framework outperforms the well-established theory. In **QR** framework, the matrix of the deformation gradient is decomposed into an orthogonal rotation tensor $\boldsymbol{\mathcal{R}}$ that transforms the coordinate system such that the matrix of the deformation gradient takes on the form of an upper-triangular matrix, $\boldsymbol{\mathcal{U}}$, called the Laplace stretch. The diagonal terms of the Laplace stretch represent extensions of the sides of a representative cube placed along the new base vectors whereas the off-diagonal terms represent the extent of shear in specific planes as will be discussed in Section 2. Imposing appropriate thermodynamic restrictions, the Laplace stretch can be *uniquely* decomposed into its elastic and plastic parts owing to the group property of a set of upper-triangular matrices. In addition, this decomposition also leads to a simplified understanding of the underlying deformation processes. For example, the first diagonal term a is related to its components through $a = a^e\,a^p$. The physical interpretation of this relation can be easily understood from the deformation of a representative cube. When a total deformation (containing both elastic and plastic parts) is applied on the cube, its side along one of the new base vectors is extended by an amount of a. Now an elastic unloading will lead to a removal of the elastic part of a, thus shortening the side by an amount of a^e; the remaining part a^p thus represents the permanent stretch on that side caused by a plastic deformation. This interpretation is missing from the traditional theory since the tangent maps $\mathbf{F}^e$ and $\mathbf{F}^p$ are not integrable and thus, do not produce a deformation map. The physical meaning of **QR** kinematics is particularly useful in plasticity because of its ability to measure the kinematic quantities via experiments. Although it has been proposed that the elastic and plastic parts of a deformation gradient can, for example, be measured by performing electron backscatter diffraction (EBSD) (Jiang et al., 2016), such propositions are flawed from a theoretical point of view because the elastic and inelastic parts of a deformation gradient are not compatible, unlike the total deformation gradient itself. Therefore, in general, it is not possible to define a deformation map between the reference and intermediate or the intermediate and current configurations. On the other hand, using the same experimental techniques, one can measure the plastic (or elastic) part of Laplace stretch $\boldsymbol{\mathcal{U}}^p$ (or $\boldsymbol{\mathcal{U}}^e$), at least up to a homogeneous rotation field. A method to determine $\boldsymbol{\mathcal{U}}^p$ has been delineated in Paul and Freed (2020a). Because of the advantages that an elastic–plastic decomposition of Laplace stretch has over other traditional decomposition, Paul and Freed (2021) developed a constitutive model for isotropic, elastic perfectly-plastic materials using scalar, conjugate, stress/strain, base pairs. The use of conjugate, stress/strain, base pairs helps one to overcome an issue with the parametrization of a material model owing to a strong covariance between the traditionally used tensor invariants, as pointed out by Criscione (2004). Needless to say, the elastic–plastic **QR** kinematics also entail the computational advantages as mentioned earlier.

Although a constitutive model based on macroscopic kinematic variables (*e.g.*, the model developed in Paul and Freed (2021)) is able to predict the overall response of a material, such a model will perform poorly whenever the microstructure of a material evolves along a path of deformation. To overcome this issue, one typically introduces internal state variables to represent macroscopic manifestations of its microstructural properties, cf. Coleman and Gurtin (1967), Rice (1971), Dafalias (1987), and Jirásek and Rolshoven (2009). In this paper, we extend our previously developed constitutive model by incorporating internal state variables into it. Typically, plastically-induced anisotropy is represented through an evolving, tensor-valued, internal, state variable, *e.g.*, Giessen (1989a,b, 1991). In our model, however, anisotropy enters into the constitutive model through certain scalar-valued anisotropy parameters. To incorporate plastically-induced anisotropy, it is sufficient that these parameters be considered as kinematic variables whose evolution equations are derived from an appropriate criterion.

This paper is organized as follows. In Section 2, **QR** kinematics and the constitutive relations based upon this kinematics are reviewed, and pertinent topics for subsequent developments are discussed in detail. To incorporate an internal state variable into a constitutive model, one needs to define an appropriate co-rotational rate for this variable. This is discussed in Section 3. In Section 4, selected internal state variables are incorporated into the authors' constitutive model. The developed model captures an evolving anisotropy during a plastic deformation process. Two example problems have been considered to demonstrate the procedure: *(i)* a Prager kinematic hardening with isotropic behavior throughout the deformation, and *(ii)* an evolving anisotropy whose contribution to the dissipation is assumed. Finally, the results are summarized and the paper is drawn to conclusion. Evolution equations governing our internal state variables are derived in the appendix.

2. Preliminaries

2.1. QR kinematics

Let us consider a simply-connected body $\mathcal{B}$ with a material particle $\mathcal{P}$ in it. Let $\mathbf{X}$ and $\mathbf{x}$ denote the position vector of this particle in an undeformed (reference) configuration of the body, *i.e.*, $\kappa_r(\mathcal{B})$, and its deformed (current) configuration, viz., $\kappa_t(\mathcal{B})$. The motion of a body is defined as $\mathbf{x}(\mathbf{X},t) = \mathcal{X}(\mathbf{X},t)$. The deformation gradient $\mathbf{F}$ is a homeomorphism that takes a tangent vector $d\mathbf{X}$ from the reference configuration $\kappa_r(\mathcal{B})$ and places it into the tangent space of the current configuration $\kappa_t(\mathcal{B})$. The deformation gradient is written as

$$\mathbf{F} = \frac{\partial \mathcal{X}(\mathbf{X},t)}{\partial \mathbf{X}}. \tag{1}$$

[1] In fact, the **QR** framework fits well with the multiple natural configurations framework, proposed by Rajagopal and Srinivasa (1998a) which also assumes the existence of a stress free natural configuration obtained by an instantaneous elastic unloading from the current configuration of the body.

We apply a Gram–Schmidt procedure to the matrix of a deformation gradient. This process decomposes it into an orthogonal rotation $\boldsymbol{\mathcal{R}}$ and an upper-triangular matrix $\boldsymbol{\mathcal{U}}$ called the Laplace stretch. Freed et al. (2019) In this procedure, a particular coordinate direction and its adjoining coordinate plane need to be specified. Given such a specification, a new set of base vectors is obtained from these specified bases though Laplace's technique of successive orthogonal projections. This coordinate direction and this coordinate plane are, in general, not known *a priori*, and as such, they can become a source for ambiguity regarding a representation for Laplace stretch. To avoid this issue, we follow a strategy put forward by Paul et al. (2021) to re-index the matrix of a deformation gradient $\mathbf{F}$ through a permutation of its base vectors, where selection of an appropriate permutation matrix is based upon the current state of deformation in a body. The Gram–Schmidt procedure is then applied on this re-indexed deformation gradient $\boldsymbol{\mathcal{F}}$, resulting in

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{U}} \quad \text{where} \quad [\mathcal{U}_{ij}] = \begin{bmatrix} a & a\gamma & a\beta \\ 0 & b & b\alpha \\ 0 & 0 & c \end{bmatrix}. \tag{2}$$

The inverse to the rotation tensor, *i.e.*, $\boldsymbol{\mathcal{R}}^T$, rotates an Eulerian set of bases into a new set of bases $\{\vec{\boldsymbol{\mathcal{E}}}_i\}$. The individual components of Laplace stretch have direct, physical meaning in this new set of bases in contrast to that of the right stretch tensor $\mathbf{U}$, obtained from a polar decomposition of the deformation gradient. Specifically, the diagonal elements of Laplace stretch, *i.e.*, a, b and c, represent elongations of the sides of a representative cube along the base vectors in this physical frame of reference; whereas, the off-diagonal terms, viz., $a\gamma, a\beta$ and $b\alpha$, represent the extents of shear acting across specific coordinate planes along the directions of different base vectors (Srinivasa, 2012, § 2). Because of this, $\{\vec{\boldsymbol{\mathcal{E}}}_i\}$ are termed the bases for a physical frame of reference. The Laplace stretch can be further decomposed into two shearing motions, followed by an extensional motion along the directions $\{\vec{\boldsymbol{\mathcal{E}}}_i\}$, $i = 1, 2, 3$, thereby resulting in an Iwasawa (1949) decomposition of Laplace stretch. This decomposition is given by

$$[\mathcal{U}_{ij}] = \underbrace{\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix}}_{\mathcal{U}^{\alpha\beta}} \underbrace{\begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathcal{U}^{\gamma}}. \tag{3}$$

The extensional matrix Λ can be further decomposed into a dilatational and three modes of squeeze, *i.e.*,

$$[\Lambda_{ij}] = \underbrace{\sqrt[3]{abc}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{dilatation}} \underbrace{\begin{bmatrix} \sqrt[3]{a/b} & 0 & 0 \\ 0 & \sqrt[3]{b/a} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{1-2 planar squeeze}} \times \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt[3]{b/c} & 0 \\ 0 & 0 & \sqrt[3]{c/b} \end{bmatrix}}_{\text{2-3 planar squeeze}} \underbrace{\begin{bmatrix} \sqrt[3]{a/c} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt[3]{c/a} \end{bmatrix}}_{\text{3-1 planar squeeze}}. \tag{4}$$

Therefore, the Laplace stretch is essentially decomposed into seven modes of deformation: one dilatation mode, three squeeze modes, and three shear modes. Note that not all of these deformation modes are independent. Specifically, the three squeeze modes are dependent on each other. The decomposition of the Laplace stretch into different modes is useful in defining the corresponding scalar, conjugate stress/strain base pairs. The scalar nature of the kinematic and kinetic variables helps in the ease of implementation and reduces computational efforts as compared to their tensorial counterparts or their invariants, used in the traditional approach.

The **QR** kinematics have been extended to elasto-plasticity by Freed et al. (2019). Using the property that the set of all invertible, upper-triangular matrices form a group under multiplication, they decomposed the total Laplace stretch into its elastic and plastic parts, denoted by $\boldsymbol{\mathcal{U}}^e$ and $\boldsymbol{\mathcal{U}}^p$, respectively. Needless to mention that both $\boldsymbol{\mathcal{U}}^e$ and $\boldsymbol{\mathcal{U}}^p$ are upper-triangular matrices. The components of these matrices are given as

$$[\mathcal{U}^e_{ij}] = \begin{bmatrix} a^e & a^e\gamma^e & a^e\beta^e \\ 0 & b^e & b^e\alpha^e \\ 0 & 0 & c^e \end{bmatrix} \quad \text{and} \quad [\mathcal{U}^p_{ij}] = \begin{bmatrix} a^p & a^p\gamma^p & a^p\beta^p \\ 0 & b^p & b^p\alpha^p \\ 0 & 0 & c^p \end{bmatrix}. \tag{5}$$

Components of the total Laplace stretch $\boldsymbol{\mathcal{U}}$ and its elastic and plastic parts $\boldsymbol{\mathcal{U}}^e$ and $\boldsymbol{\mathcal{U}}^p$ are therefore related through

$$\begin{aligned} a^e &= a/a^p; \qquad \alpha^e = b^p(\alpha - \alpha^p)/c^p; \\ b^e &= b/b^p; \qquad \beta^e = a^p\left[(\beta - \beta^p) - \alpha^p(\gamma - \gamma^p)\right]/c^p; \\ c^e &= c/c^p; \qquad \gamma^e = a^p(\gamma - \gamma^p)/b^p. \end{aligned} \tag{6}$$

It is important to note that the deformation of a body in all six of its degrees of freedom is *completely* specified by the components of Laplace stretch, whereas the rotation $\boldsymbol{\mathcal{R}}$ plays an important role in coordinate transformation. Unlike the traditional Kröner–Lee decomposition, here a decomposition of the rotation tensor into its elastic and plastic parts is not necessary in order to obtain the elastic and plastic parts of the Laplace stretch (Freed et al., 2019). Only the latter is useful when constructing constitutive models (Paul and Freed, 2021). It is instructive to consider the intermediate configuration of a body subjected only to a plastic deformation. Such a configuration is, in general, non-Euclidean and is obtained through an elastic unloading of a deformed body. In this configuration, the rotation tensor $\boldsymbol{\mathcal{R}}$ is comprised only of its plastic component, $\boldsymbol{\mathcal{R}}^p$.

To extend the **QR** kinematics to elasto-plasticity, a different approach was adopted by Ghosh and Srinivasa (2014). In this paper, a **QR** decomposition was performed on the plastic deformation gradient, obtained from a Kröner–Lee decomposition, *i.e.*, $\mathbf{F} = \mathbf{F}^e\,\mathbf{F}^p$ with $\mathbf{F}^p = \boldsymbol{\mathcal{R}}^p\,\boldsymbol{\mathcal{U}}^p$. Since the components of the plastic Laplace stretch represent the plastic deformation of a body in all six degrees of freedom, $\boldsymbol{\mathcal{U}}^p$ obtained from both these approaches ought to be the same. Now comparing the elastic–plastic **QR** decomposition with the traditional $\mathbf{F} = \mathbf{F}^e\,\mathbf{F}^p$ and defining $\boldsymbol{\mathcal{R}}^e := \boldsymbol{\mathcal{R}}^{p^T}\,\boldsymbol{\mathcal{R}}$, one can establish a relationship between Lee's elastic deformation gradient, $\mathbf{F}^e$ and the elastic Laplace stretch $\boldsymbol{\mathcal{U}}^e$ as

$$\mathbf{F} = \mathbf{F}^e\,\mathbf{F}^p = \boldsymbol{\mathcal{R}}\,\boldsymbol{\mathcal{U}}^e\,\boldsymbol{\mathcal{U}}^p \implies \mathbf{F}^e = \boldsymbol{\mathcal{R}}^p\left(\boldsymbol{\mathcal{R}}^e\,\boldsymbol{\mathcal{U}}^e\right)\boldsymbol{\mathcal{R}}^{p^T}. \tag{7}$$

Thus, the total deformation gradient can be written as

$$\mathbf{F} = \boldsymbol{\mathcal{R}}^p\,\boldsymbol{\mathcal{R}}^e\,\boldsymbol{\mathcal{U}}^e\,\boldsymbol{\mathcal{U}}^p. \tag{8}$$

The relevant configurations and the tangent maps involved in **QR** decomposition and Kröner–Lee decomposition and their relationships are shown in Fig. 1. Now let us consider a purely plastic deformation where $\mathbf{F} = \mathbf{F}^p$. Clearly, in this case, the elastic deformation gradient $\mathbf{F}^e$ as well as $\boldsymbol{\mathcal{R}}^e$ and $\boldsymbol{\mathcal{U}}^e$ are all unit second-order tensors. Using this condition on eqn. (8), one can easily obtain $\mathbf{F}^p = \boldsymbol{\mathcal{R}}^p\,\boldsymbol{\mathcal{U}}^p$. Since the **QR** decomposition of a non-singular matrix is unique (*i.e.*, since $\det(\mathbf{F}^p) \neq 0$), our plastic Laplace stretch and the corresponding plastic rotation tensor is equivalent to the ones found from a Gram–Schmidt factorization of the plastic deformation gradient $\mathbf{F}^p$, proposed in Ghosh and Srinivasa (2014). Note that although the elastic–plastic decomposition of Laplace stretch is unique owing to the group property of upper-triangular matrices, the same cannot be said about the decomposition $\mathbf{F}^p = \boldsymbol{\mathcal{R}}^p\boldsymbol{\mathcal{U}}^p$. This is due to the fact that the plastic deformation gradient, obtained from a Kröner–Lee decomposition, is not unique. Moreover, it is easy to understand that this non-uniqueness can be attributed to the plastic rotation tensor $\boldsymbol{\mathcal{R}}^p$.

At this juncture, it is important to understand the physical meanings that $\boldsymbol{\mathcal{R}}^p$ and $\boldsymbol{\mathcal{U}}^p$ carry. If a body is subjected to an elastic unloading from its current (deformed) configuration κ_t, it attains an intermediate configuration κ_p. Because $\boldsymbol{\mathcal{U}}^p$ can be obtained from a **QR** decomposition of the matrix of $\mathbf{F}^p$, the plastic deformation of a body in all six degrees of freedom is completely specified by the plastic Laplace stretch $\boldsymbol{\mathcal{U}}^p$ in a

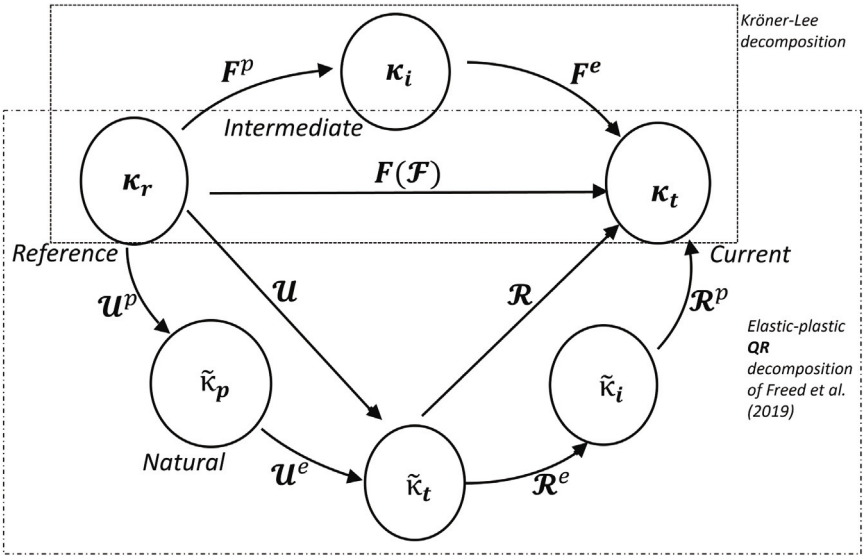

Fig. 1. The configurations involved in Kröner–Lee decomposition and **QR** decomposition and the associated tangent maps.

certain set of bases that is determined by the rotation $\mathcal{R}^p$. Let $\tilde{\kappa}_p$ denote another intermediate configuration in which the body is subjected to only the plastic part of Laplace stretch $\mathcal{U}^p$. The configurations κ_p and $\tilde{\kappa}_p$ are related through the plastic rotation $\mathcal{R}^p$. Clearly, like the total rotation tensor $\mathcal{R}$, its plastic counterpart also takes part in a coordinate transformation. In view of the physical interpretations of $\mathcal{U}^p$ and $\mathcal{R}^p$, one can easily realize a correspondence between the intermediate configuration $\tilde{\kappa}_p$ and Mandel (1971, 1973) isoclinic configuration in the sense that both these configurations represent the substructure of the material (*i.e.*, a macroscopic manifestation of the material microstructure). Investigation of the roles, played by the intermediate configuration $\tilde{\kappa}_p$ as a material substructure, is beyond the scope of this paper and will be discussed in subsequent papers in connection with studying plastic spin within our framework.

2.2. Constitutive model

Typically, constitutive models are developed using tensor invariants whenever a polar decomposition of the deformation gradient is adopted. In these models, the strain energy function W is often assumed to be a function of the invariants of relevant kinematic quantities, for example, the principal invariants of $\mathbf{C}$, denoted by I_1 and I_2. However, the widely used invariant theory is not without its drawbacks. Criscione (2004) showed that a strong covariance exists between these tensor invariants in the sense that the inner product of the response terms (*i.e.*, $\partial W/\partial I_1$ or $\partial W/\partial I_2$) turns out to be nearly equal to the product of their individual magnitudes. This co-variance between the invariants significantly magnifies the experimental error. Recently, a direct variance between the invariants has also been reported in the context of fiber reinforced elastomers (Chatterjee et al., 2021). Clearly, the covariance between the invariants hinders one's ability to parametrize the material model from an experimenter's standpoint and the orthogonality of a suitable strain measure has shown to be necessary to reduce error propagation. To overcome this issue, alternative sets of invariants (Criscione et al., 2000; Criscione, 2004) have been proposed that restrict this covariance to a minimum. As mentioned by Srinivasa (2012), the **QR** framework retains the necessary qualities (*i.e.*, exhibit minimum covariance between kinematic and response terms) from an experimental standpoint while it is simpler and easy to implement. Keeping this in mind, Freed et al. (2016, 2017), Freed (2017) developed constitutive models using scalar, conjugate, stress/strain, base pairs for isotropic and anisotropic elastic materials. In these works, a constitutive formulation was derived by deconstructing the stress power at a material point into its different modes of deformation. For the total Laplace stretch, in view of Eqs. (3) and (4), one can define the strain attributes as

$$\delta = \tfrac{1}{3}\ln(abc); \quad \varepsilon_1 = \tfrac{1}{3}\ln(a/b); \quad \varepsilon_2 = \tfrac{1}{3}\ln(b/c); \quad \varepsilon_3 = \tfrac{1}{3}\ln(c/a); \quad \gamma_1 = \alpha; \quad \gamma_2 = \beta; \quad \gamma_3 = \gamma \tag{9}$$

where δ is the volumetric strain, and ε_i and γ_i are the squeeze and shear strains, respectively.

The stress power per unit reference volume at a material point can be written as

$$\dot{W} = \mathrm{tr}(\mathbf{S}\dot{\mathbf{E}}) = J\,\mathrm{tr}(\mathbf{T}\mathbf{L}) \tag{10}$$

where $\mathbf{S}$ and $\mathbf{T}$ are the symmetric, second Piola–Kirchhoff and Cauchy stress, respectively and $\mathbf{E}$ is the Green strain. $J = \det(\mathbf{F})$ denotes the Jacobian of the deformation map. Let us now define a velocity gradient associated with the Laplace stretch as $\boldsymbol{\mathcal{L}} := \dot{\mathcal{U}}\mathcal{U}^{-1}$. A routine calculation using Eq. (2) shows that the velocity gradient $\mathbf{L} := \dot{\mathbf{F}}\mathbf{F}^{-1}$ is related to $\boldsymbol{\mathcal{L}}$ through

$$\mathbf{L} = \dot{\mathcal{R}}\mathcal{R}^T + \mathcal{R}\boldsymbol{\mathcal{L}}\mathcal{R}^T. \tag{11}$$

Needless to say that the spin tensor $\dot{\mathcal{R}}\mathcal{R}^T$ is anti-symmetric. From the second part of Eq. (10), the stress power can now be written as

$$\dot{W} = J\,\mathrm{tr}(\mathbf{T}\mathbf{L}) = J\,\mathrm{tr}\left(\mathbf{T}(\dot{\mathcal{R}}\mathcal{R}^T + \mathcal{R}\boldsymbol{\mathcal{L}}\mathcal{R}^T)\right) = J\,\mathrm{tr}\left(\mathbf{T}\mathcal{R}\boldsymbol{\mathcal{L}}\mathcal{R}^T\right) = \mathrm{tr}(\boldsymbol{\mathcal{S}}\boldsymbol{\mathcal{L}}) \tag{12}$$

where $\boldsymbol{\mathcal{S}}$ is the Kirchhoff stress in our physical frame of reference, which is related to the Cauchy stress $\mathbf{T}$ through the relation $\boldsymbol{\mathcal{S}} := J\mathcal{R}^T\mathbf{T}\mathcal{R}$.

In terms of the strain attributes defined in Eq. (9) and their thermodynamic conjugates (see Freed (2017) (Freed, 2017, § 3,4) for a detailed derivation of the conjugate stress/strain base pairs), the stress power can be expressed as

$$\dot{W} = \pi\dot{\delta} + \sum_{i=1}^{3}\left(\sigma_i\dot{\varepsilon}_i + \tau_i\dot{\gamma}_i\right) \tag{13}$$

where stresses π, σ_i and τ_i are the respective thermodynamic conjugates to strains δ, ε_i and γ_i. In terms of the stress components from $\boldsymbol{\mathcal{S}}$, these stress attributes can be written as

$$\pi = \mathcal{S}_{11} + \mathcal{S}_{22} + \mathcal{S}_{33}; \quad \sigma_1 = \mathcal{S}_{11} - \mathcal{S}_{22}; \quad \sigma_2 = \mathcal{S}_{22} - \mathcal{S}_{33}; \quad \sigma_3 = \mathcal{S}_{33} - \mathcal{S}_{11};$$
$$\tau_1 = \frac{b}{c}\mathcal{S}_{12}; \quad \tau_2 = \frac{a}{c}\mathcal{S}_{13}; \quad \tau_3 = \frac{a}{b}\mathcal{S}_{12} - \alpha\frac{a}{c}\mathcal{S}_{13}$$

(14)

where π is a pressure, σ_i is the ith normal stress difference, and τ_i is the ith shear stress, $i = 1, 2, 3$. Note that out of these seven modes of deformation, only six are independent. Specifically, a coupling exists between the three squeeze modes that leads to $\varepsilon_3 = -(\varepsilon_1 + \varepsilon_2)$ and $\sigma_3 = -(\sigma_1 + \sigma_2)$. Although such a coupling between the squeeze modes is not desirable, it seems inevitable in the present theory. In fact, as mentioned earlier, covariance exists between the invariants when a traditional constitutive assumption is used. In the alternative constitutive models, developed to address this particular issue such as Criscione et al. (2000), Criscione and Hunter (2003), Criscione (2004), a covariance is also present among the kinematic variables, but it is restricted to a minimum. The same is applicable for the proposed constitutive model as pointed out by Freed (2017).

Proof of frame invariance: Let us consider two observers with reference frame $\mathcal{O}$ and $\mathcal{O}^*$ measuring the deformation of a body. We further assume that the frames $\mathcal{O}$ and $\mathcal{O}^*$ coincides with each other at a time instant when the configuration of the body is the reference configuration. Therefore, the position vector of a particle in the reference configuration of the body is measured as $\mathbf{X}$ from both the reference frames. Now the motion of the body, observed from $\mathcal{O}$ and $\mathcal{O}^*$ are related through

$$\boldsymbol{\mathcal{X}}^*(\mathbf{X}, t^*) = \mathbf{Q}(t)\,\boldsymbol{\mathcal{X}}(\mathbf{X}, t) + \mathbf{d}(t) \tag{15}$$

where $\mathbf{Q}(t) \in \mathrm{Orth}^+$, $t^* = t - a$ with $\mathbf{Q}(0) = \mathbf{I}$ and $\mathbf{d}(0) = 0$. From eqns. (1) and (15), it is obvious that $\mathbf{F}^* = \mathbf{Q}\,\mathbf{F}$. In Section 2, the Laplace stretch $\boldsymbol{\mathcal{U}}$ was obtained from a Gram–Schmidt factorization of the deformation gradient $\mathbf{F}$. Alternatively, the components of Laplace stretch in a given coordinate system can also be found from a Cholesky factorization of the right Cauchy–Green tensor $\mathbf{C} := \mathbf{F}^T\mathbf{F}$ (Srinivasa, 2012) as

$$[\mathcal{U}_{ij}] = \begin{bmatrix} \sqrt{C_{11}} & \dfrac{C_{12}}{\mathcal{U}_{11}} & \dfrac{C_{13}}{\mathcal{U}_{11}} \\ 0 & \sqrt{C_{22} - \mathcal{U}_{12}^2} & \dfrac{C_{23} - \mathcal{U}_{12}\,\mathcal{U}_{13}}{\mathcal{U}_{22}} \\ 0 & 0 & \sqrt{C_{33} - \mathcal{U}_{13}^2 - \mathcal{U}_{23}^2} \end{bmatrix}. \tag{16}$$

The group property of the set of upper-triangular matrices ensures that this decomposition is unique for a given right Cauchy–green tensor, $\mathbf{C}$. From the relation between the deformation gradients $\mathbf{F}$ and $\mathbf{F}^*$ and the definition of the right Cauchy–Green tensor, one can easily observe that the right Cauchy–Green tensors are the same when measured in the reference frames $\mathcal{O}$ and $\mathcal{O}^*$, *i.e.*, $\mathbf{C}^* = \mathbf{C}$. In view of eqn. (16), it is obvious that the same relationship holds for the Laplace stretches, *i.e.*, $\boldsymbol{\mathcal{U}}^* = \boldsymbol{\mathcal{U}}$. The strain attributes defined in Eq. (9), being simple algebraic functions of the components of the Laplace stretch, are also going to be the same when measured from the two reference frames, *i.e.*, $\delta^* = \delta$, $\varepsilon_i^* = \varepsilon_i$ and $\gamma_i^* = \gamma_i$, $i = 1, 2, 3$. Since the strain attributes are scalar quantities, they are frame-invariant.

To show the frame invariance of the stress attributes, we shall first assume that the traction vector is objective. This assumption results in the frame invariance of the Cauchy stress tensor, *i.e.*, $\mathbf{T}^* = \mathbf{Q}\,\mathbf{T}\,\mathbf{Q}^T$. Since, $\mathbf{F}^* = \mathbf{Q}\,\mathbf{F}$, it can be easily shown that the corresponding rotation tensors are related through $\boldsymbol{\mathcal{R}}^* = \mathbf{Q}\,\boldsymbol{\mathcal{R}}$. Therefore, the Kirchhoff stress in our physical frame of reference measured from $\mathcal{O}^*$ can be written as

$$\boldsymbol{\mathcal{S}}^* = J^*\boldsymbol{\mathcal{R}}^{*\mathcal{T}}\,\mathbf{T}^*\,\boldsymbol{\mathcal{R}}^* = J\,(\mathbf{Q}\,\boldsymbol{\mathcal{R}})^T\,(\mathbf{Q}\,\mathbf{T}\,\mathbf{Q}^T)\,(\mathbf{Q}\,\boldsymbol{\mathcal{R}}) = J\,\boldsymbol{\mathcal{R}}^{\mathcal{T}}\,\mathbf{T}\,\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{S}}. \tag{17}$$

From the definition of the stress attributes in eqn. (14), one can write $\pi^* = \pi$, $\sigma_i^* = \sigma_i$ and $\tau_i^* = \tau_i$. Since the stress attributes are scalars, they are frame invariant.

A relationship between these stress and strain attributes can be established from a thermodynamic consideration. Needless to say that such relationships depend upon the behavior exhibited by the material under consideration. For convenience in notation, let us define three lists of variables consisting of the stress and strain attributes and their rates as

$$l_{\mathcal{U}} := \{\ \delta \quad \varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3 \quad \gamma_1 \quad \gamma_2 \quad \gamma_3\ \}; \tag{18a}$$

$$l_{\dot{\mathcal{U}}} := \{\ \dot{\delta} \quad \dot{\varepsilon}_1 \quad \dot{\varepsilon}_2 \quad \dot{\varepsilon}_3 \quad \dot{\gamma}_1 \quad \dot{\gamma}_2 \quad \dot{\gamma}_3\ \}; \tag{18b}$$

$$l_S = \{\ \pi \quad \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \tau_1 \quad \tau_2 \quad \tau_3\ \}. \tag{18c}$$

Similar to Eq. (9), for an elastic–plastic material, one can define the elastic and plastic strain attributes as

$$\delta^e = \tfrac{1}{3}\ln(a^e b^e c^e);\quad \varepsilon_1^e = \tfrac{1}{3}\ln(a^e/b^e);\quad \varepsilon_2^e = \tfrac{1}{3}\ln(b^e/c^e);\quad \varepsilon_3^e = \tfrac{1}{3}\ln(c^e/a^e);$$
$$\gamma_1^e = \alpha^e;\quad \gamma_2^e = \beta^e;\quad \gamma_3^e = \gamma^e \tag{19 a}$$

and

$$\delta^p = \tfrac{1}{3}\ln(a^p b^p c^p);\quad \varepsilon_1^p = \tfrac{1}{3}\ln(a^p/b^p);\quad \varepsilon_2^p = \tfrac{1}{3}\ln(b^p/c^p);\quad \varepsilon_3^p = \tfrac{1}{3}\ln(a^p/c^p);$$
$$\gamma_1^p = \alpha^p;\quad \gamma_2^p = \beta^p;\quad \gamma_3^p = \gamma^p. \tag{19 b}$$

These elastic and plastic strain attributes are related to the total strain attributes via

$$\begin{aligned} \delta &= \delta^e + \delta^p, \\ \varepsilon_1 &= \varepsilon_1^e + \varepsilon_1^p, \\ \varepsilon_2 &= \varepsilon_2^e + \varepsilon_2^p, \\ \varepsilon_3 &= \varepsilon_3^e + \varepsilon_3^p, \\ \gamma_1 &= (c^p/b^p)\,\gamma_1^e + \gamma_1^p, \\ \gamma_2 &= (c^p/a^p)\,\gamma_2^e + \gamma_2^p + \gamma_1^p\,(\gamma_3 - \gamma_3^p), \\ \gamma_3 &= (b^p/a^p)\,\gamma_3^e + \gamma_3^p. \end{aligned} \tag{20}$$

For a convenience in notation, let us define a list of variables $l_{\mathcal{U}^p}$ consisting of the plastic strain attributes. This list of variables is defined as

$$l_{\mathcal{U}^p} := \{\ \delta^p \quad \varepsilon_1^p \quad \varepsilon_2^p \quad \varepsilon_3^p \quad \gamma_1^p \quad \gamma_2^p \quad \gamma_3^p\ \}. \tag{21}$$

It is interesting to note that in this framework, an anisotropic material response does not enter into the constitutive model directly through the material parameters (*i.e.*, through symmetry of the stiffness or compliance matrix). Instead, the anisotropy is enfolded in the encoding/decoding map that relates components of the velocity gradient $\boldsymbol{\mathcal{L}} := \dot{\boldsymbol{\mathcal{U}}}\,\boldsymbol{\mathcal{U}}^{-1}$ with our strain rate attributes, and components of the Kirchhoff stress $\boldsymbol{\mathcal{S}}$ with our stress attributes. For an anisotropic elastic materials, a relationship between the components of $\boldsymbol{\mathcal{L}}$ and the strain rate attributes is given as

$$\begin{Bmatrix} \dot{\delta} \\ \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{Bmatrix} = \begin{bmatrix} vw/3u & uw/3v & uv/3w & 0 & 0 & 0 \\ vw/3u & -uw/3v & 0 & 0 & 0 & 0 \\ 0 & uw/3v & -uv/3w & 0 & 0 & 0 \\ 0 & 0 & 0 & c/b & 0 & 0 \\ 0 & 0 & 0 & 0 & c/a & b\gamma_1/a \\ 0 & 0 & 0 & 0 & 0 & b/a \end{bmatrix} \begin{Bmatrix} \mathcal{L}_{11} \\ \mathcal{L}_{22} \\ \mathcal{L}_{33} \\ \mathcal{L}_{23} \\ \mathcal{L}_{13} \\ \mathcal{L}_{12} \end{Bmatrix} \tag{22}$$

whereas the stress attributes are related to the components of $\boldsymbol{\mathcal{S}}$ via

$$\begin{Bmatrix} \pi \\ \sigma_1 \\ \sigma_2 \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{Bmatrix} = \begin{bmatrix} u/vw & v/uw & w/uv & 0 & 0 & 0 \\ u/vw & -v/uw & 0 & 0 & 0 & 0 \\ 0 & v/uw & -w/uv & 0 & 0 & 0 \\ 0 & 0 & 0 & b/c & 0 & 0 \\ 0 & 0 & 0 & 0 & a/c & 0 \\ 0 & 0 & 0 & 0 & -a\gamma_1/c & a/b \end{bmatrix} \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{23} \\ S_{13} \\ S_{12} \end{Bmatrix} \tag{23}$$

where u, v and w are anisotropy parameters representing the strength on anisotropy along the directions $\vec{\boldsymbol{\varepsilon}}_1$, $\vec{\boldsymbol{\varepsilon}}_2$ and $\vec{\boldsymbol{\varepsilon}}_3$, respectively, relative to the other directions. For an isotropic material, each of these parameters equals one. The reader is referred to Freed et al. (2016) and Freed (2017) for a detailed derivation of Eqs. (22) and (23).

3. Internal state variables and their co-rotational rates

While constructing constitutive models for elastic–plastic materials, it is a common practice to consider only quantities representing the macroscopic deformation of a body, *e.g.*, the Laplace stretch $\boldsymbol{\mathcal{U}}$ and its plastic part $\boldsymbol{\mathcal{U}}^p$ as kinematic variables, along with their corresponding work conjugates. However, in these models, it is not possible to keep track of an evolution of the underlying microstructural properties of a material with these kinematic variables. Therefore, although these models work well for isotropic materials, they are unable to capture material responses that exhibit evolving microstructural properties, such as plastically-induced anisotropy. In order to resolve this issue, internal state variables are typically used that act as a macroscopic manifestation of these microstructural features. These internal variables can be scalars, vectors or tensors, and can represent both kinematic and kinetic variables, such as a back stress, orientation of the lattice vectors, etc.

Let $\mathbf{a}_i, i = 1, 2, ..., n$ denote a set of n number of internal variables in the current configuration κ_t. We assume that these internal state variables follow a standard pull back or push forward operation since the base vectors are related through the associated tangent maps. Before going into a detailed discussion on how these internal variables are incorporated in our framework, let us first discuss their physical significance using a traditional Kröner–Lee decomposition. Upon elastic unloading, this set of variables is pulled back into configuration κ_p and is denoted by $\mathbf{A}_i$. Sets $\mathbf{a}_i$ and $\mathbf{A}_i$ are related through an inverse of the elastic deformation gradient and its transpose. However, the specific relation depends on the nature of a particular internal variable. For instance, if $\mathbf{a}_i$ is a tensor-valued internal variable, then $\mathbf{A}_i$ can be obtained as $\mathbf{A}_i = \det(\mathbf{F}^e)\,\mathbf{F}^{e^{-1}}\,\mathbf{a}_i\,\mathbf{F}^{e^{-T}}$. Because $\mathbf{A}_i$ represents a macroscopic manifestation of the microstructural changes, the material response must also depend on these internal variables. Here we first consider the internal variable to be a tensor-valued kinematic variable since we derive the constitutive model by allowing the kinematic variables (*i.e.*, the strain rate attributes) to vary (see Section 4). Internal state variable of kinetic nature are considered whenever the kinetic variables (*i.e.*, the stress attributes) are allowed to vary keeping the kinematic variables fixed. Since the kinematic and kinetic internal variables are work-conjugate to each other, the nature of the internal state variable does not pose any difficulty in deriving the constitutive model. Nevertheless one should be careful in using the right internal state variable in their derivation of a constitutive model.

It is well-known that the laws of thermodynamics are not sufficient to specify the response of an elastic–plastic material. Additionally, one needs to stipulate a more stringent criterion such as maximum plastic dissipation, maximum plastic work, Drucker's stability postulate, etc. In their constitutive model for isotropic, elastic–plastic materials, Paul and Freed (2021) used a maximum rate of dissipation criterion developed by Rajagopal and Srinivasa (1998a,b). This criterion is a generalized version of Onsager's minimum rate of dissipation criterion (Onsager, 1931) and Ziegler's normality rule (Ziegler, 1963) (cf. Rajagopal and Srinivasa (2004)). It is worth noting that although the maximum rate of dissipation criterion is applicable to a wide class of materials, this principle, along with the other criteria mentioned above, is not as fundamental as the laws of thermodynamics. Nevertheless, the use of such a criterion is ubiquitous in plasticity literature.

In this framework, the response of an elastic–plastic material is specified through constitutive assumptions arising from two functions: a stored energy function (or Helmholtz potential function)[2] denoted as ψ, and a rate of dissipation function denoted as ξ. The Helmholtz potential function specifies the elastic response of a body, measured from a fixed natural configuration $\tilde{\kappa}_p$ whereas the evolution of this natural configuration caused by microstructural change is specified through a rate of dissipation function. They further assumed that the Helmholtz potential is a function of the total Laplace stretch $\boldsymbol{\mathcal{U}}$ and its plastic part $\boldsymbol{\mathcal{U}}^p$. On the other hand, the rate of dissipation function was assumed to depend on the plastic Laplace stretch $\boldsymbol{\mathcal{U}}^p$ and its rate $\dot{\boldsymbol{\mathcal{U}}}^p$. Note that this model was developed for an isotropic material that exhibits an elastic perfectly-plastic behavior.

In order to incorporate material behavior induced by microstructural changes, one needs to also consider internal state variables as arguments of the Helmholtz potential ψ and the rate of dissipation function ξ. One particular material behavior caused by microstructural change is that of plastically-induced anisotropy. This anisotropy is exhibited at a microstructural level, and is different from the macroscopic behavior exhibited by an initially anisotropic material. The anisotropy of a material enters into our constitutive model through a particular mapping between the kinematic (*e.g.*, components of Laplace stretch) and kinetic (*e.g.*, components of the Kirchhoff stress tensor, pulled back into our physical frame of reference) quantities and their corresponding strain and stress attributes (see Section 2.2). A plastically-induced (and thus, evolving) anisotropy can be incorporated by considering certain parameters of these maps as variables.

Because the internal state variables used in our constitutive construction of an elastic–plastic material reflect a changing microstructure, an evolution equation must be specified for each $\mathbf{A}_i$ in order to keep track of its evolution, plus any change in orientation caused by plastic deformation. Specifically, an appropriate rate for each internal state variable must be specified. Recall that the physical representation of configuration $\tilde{\kappa}_p$ is that of the substructure of a material, and hence, all constitutive relations are to be formulated in this configuration. It is also worth noting that an infinitesimal fiber in the configuration κ_p is obtained by employing Lee's plastic deformation gradient, $\mathbf{F}^p$ on an infinitesimal fiber in the reference configuration of the body. Moreover, it has been shown in Section 2 that $\mathbf{F}^p$ is related to the plastic Laplace stretch via $\mathbf{F}^p = \boldsymbol{\mathcal{R}}^p\,\boldsymbol{\mathcal{U}}^p$ and this decomposition is unique. Therefore, considering a tensorial internal state variable, it must be pulled back into this configuration through the relation $\boldsymbol{\mathcal{A}}_i = \boldsymbol{\mathcal{R}}^{p^T}\,\mathbf{A}_i\,\boldsymbol{\mathcal{R}}^p$. When expressed in the set of bases $\{\tilde{\boldsymbol{\mathcal{E}}}_i\}$, it is reasonable to assume that the matrix of $\boldsymbol{\mathcal{A}}_i$ will be a full matrix. We further assume that this matrix has a non-zero determinant, *i.e.*, $\det(\boldsymbol{\mathcal{A}}_i) \neq 0$. Under these conditions, one can perform a Gram–Schmidt procedure on the matrix of $\boldsymbol{\mathcal{A}}_i$ resulting in [3]

$$\boldsymbol{\mathcal{A}}_i = \boldsymbol{\mathcal{R}}^{\mathcal{A}_i}\,\boldsymbol{\mathcal{U}}^{\mathcal{A}_i} \tag{24}$$

where $\boldsymbol{\mathcal{R}}^{\mathcal{A}_i}$ is an orthogonal matrix, and $\boldsymbol{\mathcal{U}}^{\mathcal{A}_i}$ is an upper-triangular matrix. Clearly, the upper-triangular matrix $\boldsymbol{\mathcal{U}}^{\mathcal{A}_i}$ represents the "rotation-free" part of internal state variable $\boldsymbol{\mathcal{A}}_i$ and its components are given in a new set of bases obtained through a rotation of the set of bases $\{\tilde{\boldsymbol{\mathcal{E}}}_i\}$ by $\boldsymbol{\mathcal{R}}^{\mathcal{A}_i}$. The assumption of a non-zero determinant for the matrix of $\boldsymbol{\mathcal{A}}_i$ ensures that its decomposition in Eq. (24) is unique. For the time being, we focus on the rotation part of the internal variable. Nevertheless, its counterpart $\boldsymbol{\mathcal{U}}^{\mathcal{A}_i}$ plays an essential role in constitutive formulation, and will be discussed later.

From the physical significance of a Gram–Schmidt decomposition, as discussed in Section 2.1, it is apparent that orthogonal tensor $\boldsymbol{\mathcal{R}}^{\mathcal{A}_i}$ represents a change in the orientation of internal state variable $\boldsymbol{\mathcal{A}}_i$ with respect to the bases of space $\tilde{\kappa}_p$, and hence, the substructure of this material. Therefore, a spin tensor $\boldsymbol{\Omega}^p$ defined as $\boldsymbol{\Omega}^p := \dot{\boldsymbol{\mathcal{R}}}^{\mathcal{A}_i}\,\boldsymbol{\mathcal{R}}^{\mathcal{A}_i{}^T}$

[2] This thermodynamic potential is often referred to as the Helmholtz free energy per unit mass or free energy density (Noll, 1974) or specific free energy (Truesdell and Noll, 1992) and is related to the internal energy ε, entropy η and temperature θ through a Legendre transformation viz., $\psi = \varepsilon - \eta\,\theta$.

[3] Note that this decomposition is specific to the **QR** framework. In case of a traditional framework based on a polar decomposition of the deformation gradient, typically a full second-order tensor form of the internal variable or its invariants are used for constitutive formulation.

represents the spin of the structural internal variable $\mathcal{A}_i$ taken with respect to its substructure. This spin tensor can now be defined as the plastic spin corresponding to the internal structure variable $\mathcal{A}_i$, in accordance with the terminology of Dafalias (1998).

Although it is sufficient for the plastic strain-rate attributes to use only a simple time derivative of the pertinent kinematic variables in their definitions, the same is not true for the internal state variables. It is evident from the discussion on **QR** decomposition in Section 2.1 that the matrix of $\mathcal{A}_i$ takes on the form of an upper-triangular matrix $\boldsymbol{\mathcal{U}}^{\mathcal{A}_i}$ in a *new* set of base vectors rotated from the substructure (*i.e.*, the configuration $\tilde{\kappa}_p$) by $\boldsymbol{\mathcal{R}}^{\mathcal{A}_i}$. Therefore, we must work with a suitable objective rate of the "rotation-free" part of the internal state variable $\boldsymbol{\mathcal{U}}^{\mathcal{A}_i}$ with respect to the configuration $\tilde{\kappa}_p$. Since $\boldsymbol{\mathcal{U}}^{\mathcal{A}_i}$ is defined in a co-ordinate system rotated by $\boldsymbol{\mathcal{R}}^{\mathcal{A}_i}$, it is appropriate to introduce a Lie derivative (Marsden and Hughes, 2012) of the form

$$\mathcal{L}_{\varphi^p}[\boldsymbol{\mathcal{U}}^{\mathcal{A}_i}] = \boldsymbol{\mathcal{R}}^{\mathcal{A}_i}\left[\frac{d}{dt}\left(\boldsymbol{\mathcal{R}}^{\mathcal{A}_i{}^{-1}}\,\boldsymbol{\mathcal{U}}^{\mathcal{A}_i}\,\boldsymbol{\mathcal{R}}^{\mathcal{A}_i{}^{-T}}\right)\boldsymbol{\mathcal{R}}^{\mathcal{A}_i{}^{T}}\right]. \tag{25}$$

It is worth noting that Eq. (25) is quite similar to the Green–Naghdi rate of the Kirchhoff stress. In fact, $\mathcal{L}_{\varphi^p}[\boldsymbol{\mathcal{U}}^{\mathcal{A}_i}]$ is a particularization of the Green–Naghdi rate for appropriate tensor fields in the sense of Simo and Marsden (1984). Simplifying Eq. (25), the co-rotational rate of $\boldsymbol{\mathcal{U}}^{\mathcal{A}_i}$ with respect to the plastic spin $\boldsymbol{\Omega}^p$ can be written as

$$\mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i} = \dot{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i} - \boldsymbol{\Omega}^p\,\boldsymbol{\mathcal{U}}^{\mathcal{A}_i} + \boldsymbol{\mathcal{U}}^{\mathcal{A}_i}\,\boldsymbol{\Omega}^p. \tag{26}$$

Since Lie derivatives are, by definition, frame-indifferent, the rate $\mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i}$ can now be used in our constitutive formulation. Note that unlike the other kinematic variables, $\mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i}$ is a full matrix. This causes an inevitable impediment to utilize the full potential of an upper-triangular decomposition.

4. Incorporation into constitutive model

In this section, we focus on developing a constitutive model for elastic–plastic materials that captures plastically-induced anisotropy as well as a general, tensor-valued, kinematic, internal variable. The question of which kinematic variables (plastic strain rate attributes, or the objective rate of an internal state variable) are to be used depends upon the configuration in which the constitutive relations are formulated. From the discussions in Sections 3 and 2.1, it is quite evident that the configuration $\tilde{\kappa}_p$ is of utmost importance in our framework, mainly for two reasons: (i) the components of the plastic Laplace stretch are measured in this configuration, and (ii) physically it represents a macroscopic manifestation of the material substructure. Recall that unlike the plastic deformation gradient, $\mathbf{F}^p$, arising from a multiplicative decomposition of the deformation gradient, the plastic Laplace stretch stems from a decomposition of the "rotation-free" Laplace stretch $\boldsymbol{\mathcal{U}}$. Moreover, the plastic Laplace stretch is measured in the configuration $\tilde{\kappa}_p$ which implies that the measured plastic strain rate attributes identically co-rotate with the substructure of a material. Therefore, it is reasonable to define the plastic strain rate attributes through an appropriate encoding/decoding map in a similar fashion as in Eq. (22). The plastic strain rate attributes are defined as

$$\begin{Bmatrix} \dot{\delta}^p \\ \dot{\varepsilon}_1^p \\ \dot{\varepsilon}_2^p \\ \dot{\gamma}_1^p \\ \dot{\gamma}_2^p \\ \dot{\gamma}_3^p \end{Bmatrix} = \begin{bmatrix} v w/3u & u w/3v & u v/3w & 0 & 0 & 0 \\ v w/3u & -u w/3v & 0 & 0 & 0 & 0 \\ 0 & u w/3v & -u v/3w & 0 & 0 & 0 \\ 0 & 0 & 0 & c^p/b^p & 0 & 0 \\ 0 & 0 & 0 & 0 & c^p/a^p & b^p\gamma_1^p/a^p \\ 0 & 0 & 0 & 0 & 0 & b^p/a^p \end{bmatrix} \begin{Bmatrix} \mathcal{L}_{11}^p \\ \mathcal{L}_{22}^p \\ \mathcal{L}_{33}^p \\ \mathcal{L}_{23}^p \\ \mathcal{L}_{13}^p \\ \mathcal{L}_{12}^p \end{Bmatrix} \tag{27}$$

with

$$l_{\dot{\mathcal{U}}^p} := \{\ \dot{\delta}^p \quad \dot{\varepsilon}_1^p \quad \dot{\varepsilon}_2^p \quad \dot{\varepsilon}_3^p \quad \dot{\gamma}_1^p \quad \dot{\gamma}_2^p \quad \dot{\gamma}_3^p\ \} \tag{28}$$

where $\boldsymbol{\mathcal{L}}^p := \dot{\boldsymbol{\mathcal{U}}}^p\,\boldsymbol{\mathcal{U}}^{p^{-1}}$ and $\dot{\varepsilon}_3^p = -(\dot{\varepsilon}_1^p + \dot{\varepsilon}_2^p)$. For convenience, let us replace the parameters u, v and w with n_j, $j = 1,2,3$, defined as $n_1 := u/vw, n_2 := v/uw$ and $n_3 := w/uv$. Note that although Eqs. (18a), (18b), (18c), (20) and (21) are still valid for the case of anisotropic materials, definitions of the total strain attributes and their plastic counterparts need to be revised. While determining the total and plastic strain attributes by integrating the strain rates, one must keep in mind that the parameters u, v and w (or alternatively, n_j, $j = 1,2,3$) must be considered as variables here in order to capture the development of induced anisotropy.

The upper-triangular matrix $\boldsymbol{\mathcal{U}}^{\mathcal{A}_i}$ physically represents the current state of the internal variable at a particular time instant. Moreover, due to its upper-triangular nature, it is possible to decompose this matrix similar to the decomposition of Laplace stretch in Eqs. (3) and (4) and define a list of variables containing seven scalar variables, each corresponding to a separate mode of deformation of the material substructure, that collectively represent an internal state variable in a rotated coordinate frame with respect to the material substructure. This list of variables consisting of these scalar variables is given as

$$l_{\mathcal{A}_i} = \{\ \delta^{\mathcal{A}_i} \quad \varepsilon_1^{\mathcal{A}_i} \quad \varepsilon_2^{\mathcal{A}_i} \quad \varepsilon_3^{\mathcal{A}_i} \quad \gamma_1^{\mathcal{A}_i} \quad \gamma_2^{\mathcal{A}_i} \quad \gamma_3^{\mathcal{A}_i}\ \} \tag{29}$$

where $\delta^{\mathcal{A}_i}$, $\varepsilon_j^{\mathcal{A}_i}$ and $\gamma_j^{\mathcal{A}_i}$, $j = 1,2,3$, represent attributes of the internal state variable $\mathcal{A}_i$ that correspond with dilatation, squeeze and shear of the substructure, respectively. Despite the physical meaning of components in list $l_{\mathcal{A}_i}$, and its congruence with the current theory, these scalar variables cannot be used in the constitutive formulation. Specifically, because there is no reason for the co-rotational rate of an internal state variable $\mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i}$ to be upper-triangular, and as such, it cannot be decomposed into the different modes of deformation, and thus, cannot be expressed by such a collection of scalar variables. Therefore, one must deal with tensorial variables (such as $\mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i}$) when it comes to working with tensor valued, internal, state variables like $\mathcal{A}_i$, instead of a simple time derivative of the collection of scalar variables listed in $l_{\mathcal{A}_i}$. This is a major consequence of using the co-rotational rate of an internal state variable in our theory.

Now we proceed to derive the evolution equations for the plastic strain rate attributes $l_{\dot{\mathcal{U}}^p}$, the anisotropy parameters n_j, and the internal state variables $\mathcal{A}_i$. Here, in addition to the laws of thermodynamics, we adopt a maximum rate of dissipation criterion. In our framework, the configuration $\tilde{\kappa}_p$ acts as a natural configuration from which the elastic response of the body is measured. The natural configuration itself evolves with a dissipative, plastic deformation process in the sense that the elastic unloading of a body from its current configuration leads it to occupy different natural configurations at different time instants whenever microstructural changes in the body (*e.g.*, dislocation movements) are involved. Therefore, the response of a body can be described as a family of elastic responses measured from a set of evolving natural configurations. We assume that for each natural configuration there exists a non-null elastic domain, *i.e.*, for a fixed microstructure (or natural configuration) and a given Green elastic strain $\mathbf{E}$ measured from the reference configuration, the only admissible value for the inelastic velocity gradient $\boldsymbol{\mathcal{L}}^p$ would be zero. In other words, the inverse image of $\boldsymbol{\mathcal{L}} = \mathbf{0}$ would be a non-empty elastic domain for a given microstructure and Green elastic strain Rajagopal and Srinivasa (1998b, § II.1). Therefore, we admit two functions prior to applying a maximum rate of dissipation criterion: (i) a Helmholtz potential function ψ from which the elastic response of a body for a fixed natural configuration is derived, and (ii) a dissipation function ξ representing the energy dissipated during a plastic deformation process, *i.e.*, an evolution of the natural configuration $\tilde{\kappa}_p$. For the sake of generality, throughout the constitutive formulation, we will assume that the material response is anisotropic.

Because the elastic response of a body depends upon the deformation of that body measured from its undeformed configuration κ_r and its fixed natural configuration $\tilde{\kappa}_p$, it is reasonable to assume that the

Helmholtz potential function has the form

$$\psi = \overline{\psi}(\boldsymbol{\mathcal{U}}, \boldsymbol{\mathcal{U}}^p, \overline{n}_j) = \hat{\psi}(l_{\mathcal{U}}, l_{\mathcal{U}^p}, \overline{n}_j) \tag{30}$$

where $\dot{\overline{n}}_j = n_j$, $j = 1, 2, 3$. Note that we have not considered any internal state variables $\boldsymbol{\mathcal{A}}_i$ in the argument of ψ, because the internal state variables only associate with the plastic deformation of a body. Now the elastic domain of the material for a fixed natural configuration is characterized by

$$\boldsymbol{\mathcal{L}}^p = \mathbf{0} \implies \dot{\delta}^p = \dot{\epsilon}^p_j = \dot{\gamma}^p_j = 0. \tag{31}$$

The rate of dissipation function can be determined from an isothermal energy balance equation. If $\boldsymbol{\mathcal{P}}_i$ denotes the kinetic conjugate[4] of the internal state variable $\boldsymbol{\mathcal{A}}_i$, then the rate of dissipation can be defined as

$$\xi := \dot{W} - \rho_0 \dot{\psi} - \boldsymbol{\mathcal{P}}_i : \mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i} \geq 0 \tag{32}$$

where for any two tensors $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} : \mathbf{B}$ denotes the scalar dot product such that $\mathbf{A} : \mathbf{B} = A_{ij} B_{ij}$. An assumption of non-negativity of ξ ensures that the Clausius–Duhem inequality is identically satisfied. From Eq. (32), it is evident that the dissipation function has the functional form of

$$\xi = \overline{\xi}\left(\boldsymbol{\mathcal{U}}^p, \dot{\boldsymbol{\mathcal{U}}}^p, n_j, \mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i}\right) = \hat{\xi}\left(\delta^p, \epsilon^p_j, \gamma^p_j, \dot{\delta}^p, \dot{\epsilon}^p_j, \dot{\gamma}^p_j, n_j, \mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i}\right). \tag{33}$$

Note that unlike the Helmholtz potential function ψ, the argument of the dissipation rate $\hat{\xi}$ contains both scalar as well as tensorial kinematic variables owing to the nature of the co-rotation rate of the internal state variables. Moreover, the Helmholtz potential function and the rate of dissipation function both explicitly and implicitly depend on the anisotropy parameters n_j, as they are directly related to the dilatational and squeeze strain-rate attributes. Now invoking Eq. (30) into Eq. (32), we obtain

$$\left(\pi - \rho_0 \frac{\partial \hat{\psi}}{\partial \delta}\right) \dot{\delta} + \sum_{j=1}^{3} \left[\left(\sigma_j - \rho_0 \frac{\partial \hat{\psi}}{\partial \epsilon_j}\right) \dot{\epsilon}_j + \left(\tau_j - \rho_0 \frac{\partial \hat{\psi}}{\partial \gamma_j}\right) \dot{\gamma}_j\right] = \xi + \rho_0 \frac{\partial \hat{\psi}}{\partial \delta^p} \dot{\delta}^p + \rho_0 \sum_{j=1}^{3} \left[\frac{\partial \hat{\psi}}{\partial \epsilon^p_j} \dot{\epsilon}^p_j + \rho_0 \frac{\partial \hat{\psi}}{\partial \gamma_j} \dot{\gamma}_j + \frac{\partial \hat{\psi}}{\partial \overline{n}_j} n_j\right] + \boldsymbol{\mathcal{P}}_i : \mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i}. \tag{34}$$

Let us assume that the elastic response of the body is that of a Green elastic solid. Therefore, the total stress attributes can be written in terms of derivatives of the Helmholtz potential function with respect to the total strain attributes as

$$\pi = \rho_0 \frac{\partial \hat{\psi}}{\partial \delta}, \tag{35a}$$

$$2\sigma_1 + \sigma_2 = \rho_0 \frac{\partial \hat{\psi}}{\partial \epsilon_1}, \tag{35b}$$

$$\sigma_1 + 2\sigma_2 = \rho_0 \frac{\partial \hat{\psi}}{\partial \epsilon_2}, \tag{35c}$$

$$\tau_j = \rho_0 \frac{\partial \hat{\psi}}{\partial \gamma_j}, \quad j = 1, 2, 3. \tag{35d}$$

with $\sigma_3 = -(\sigma_1 + \sigma_2)$ and $\epsilon_3 = -(\epsilon_1 + \epsilon_2)$. With this assumption describing the elastic response of a body, Eq. (34) reduces to

$$\hat{\xi} + \rho_0 \frac{\partial \hat{\psi}}{\partial \delta^p} \dot{\delta}^p + \rho_0 \sum_{j=1}^{3} \left[\frac{\partial \hat{\psi}}{\partial \epsilon^p_j} \dot{\epsilon}^p_j + \rho_0 \frac{\partial \hat{\psi}}{\partial \gamma_j} \dot{\gamma}_j + \frac{\partial \hat{\psi}}{\partial \overline{n}_j} n_j\right] + \boldsymbol{\mathcal{P}}_i : \mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i} = 0. \tag{36}$$

Now to determine evolution equations for the plastic strain rates and the internal state variables, we apply a principle of maximum rate of dissipation. According to this criterion, of all the admissible values for the plastic velocity gradient $\boldsymbol{\mathcal{L}}^p$, anisotropy parameters n_j, and co-rotational rate of the internal state variable $\mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i}$ satisfying the reduced rate of dissipation constraint (36), the ones that maximize the rate of dissipation ξ will govern the evolution of a natural configuration $\tilde{\kappa}_p$. Note that if the anisotropy parameters u, v and w were considered to be constants, *i.e.*, if evolution of anisotropy during a plastic deformation process were not to be considered, then one could carry out a maximization of ξ with respect to the variables listed in $l_{\mathcal{U}^p}$ individually, instead of the plastic velocity gradient $\boldsymbol{\mathcal{L}}^p$ or its components. However, in this case, components of the plastic velocity gradient need to be expressed in terms of the plastic strain-rate attributes and the anisotropy parameters before carrying out the optimization process. This is achieved by inverting the relation (22), which yields

$$\begin{Bmatrix} \mathcal{L}^p_{11} \\ \mathcal{L}^p_{22} \\ \mathcal{L}^p_{33} \\ \mathcal{L}^p_{23} \\ \mathcal{L}^p_{13} \\ \mathcal{L}^p_{12} \end{Bmatrix} = \begin{bmatrix} n_1 & 2n_1 & n_1 & 0 & 0 & 0 \\ n_2 & -n_2 & n_2 & 0 & 0 & 0 \\ n_3 & -n_3 & -2n_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & b^p/c^p & 0 & 0 \\ 0 & 0 & 0 & 0 & a^p/c^p & -a^p\gamma^p_1/c^p \\ 0 & 0 & 0 & 0 & 0 & a^p/b^p \end{bmatrix} \begin{Bmatrix} \dot{\delta}^p \\ \dot{\epsilon}^p_1 \\ \dot{\epsilon}^p_2 \\ \dot{\gamma}^p_1 \\ \dot{\gamma}^p_2 \\ \dot{\gamma}^p_3 \end{Bmatrix}. \tag{37}$$

Note that the relationships between the components of $\boldsymbol{\mathcal{L}}^p$ and the shear strain rates $\dot{\gamma}^p_i$ do not involve the anisotropy parameters. Therefore, for the shear modes of deformation, it is reasonable to carry out the maximization of ξ with respect to the strain rate attributes $\dot{\gamma}^p_i$. However, this cannot be done for the dilatation and squeeze modes of deformation. In these cases, an optimization process must be carried out explicitly with respect to the components of $\boldsymbol{\mathcal{L}}^p$, specifically $\boldsymbol{\mathcal{L}}^p_{11}$, $\boldsymbol{\mathcal{L}}^p_{22}$ and $\boldsymbol{\mathcal{L}}^p_{33}$. Here we have considered that the volume of a body changes during plastic deformation. Therefore, the optimization process is executed with only the reduced rate of dissipation constraint (36). In case of metal plasticity, it is often considered that the plastic deformation process is volume-preserving, *i.e.*, $\dot{\delta}^p = 0$. This condition enters into the constitutive model as an additional constraint whenever a volume-preserving motion is considered.

The maximization process can be worked out using two different methods depending on the nature of the rate of dissipation function. If ξ is assumed to be a smooth function, then the optimization process can be carried out using a standard Lagrange multiplier technique with respect to the components of $\boldsymbol{\mathcal{L}}^p$, the anisotropy parameters n_j, and a co-rotational rate of the internal state variables $\mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i}$ with ξ being the objective function and Eq. (36) acting as a constraint. A smooth rate of dissipation function is typically exhibited by materials that do not have a definite yield surface and possess creep-like behavior, often denoted as being viscoplastic. Details of this maximization process are provided in Appendix. Using a Lagrange multiplier technique to maximize ξ, we obtain

$$\frac{\partial \hat{\xi}}{\partial \dot{\delta}^p} = -\lambda \rho_0 \frac{\partial \hat{\psi}}{\partial \delta^p}; \qquad \frac{\partial \hat{\xi}}{\partial \dot{\epsilon}^p_j} = -\lambda \rho_0 \frac{\partial \hat{\psi}}{\partial \epsilon^p_j}; \qquad \frac{\partial \hat{\xi}}{\partial \dot{\gamma}^p_j} = -\lambda \rho_0 \frac{\partial \hat{\psi}}{\partial \gamma^p_j} \tag{38a}$$

$$\frac{\partial \hat{\xi}}{\partial n_j} = -\lambda \rho_0 \frac{\partial \hat{\psi}}{\partial \overline{n}_j} \tag{38b}$$

$$\frac{\partial \hat{\xi}}{\partial \mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i}} = -\lambda \boldsymbol{\mathcal{P}}_i \tag{38c}$$

where $\lambda = \overline{\lambda}/(1+\overline{\lambda})$ with $\overline{\lambda}$ being a Lagrange multiplier to be determined by substituting Eqs. (38a), (38b) and (38c) into the reduced rate of dissipation constraint (36). Thus, λ can be obtained as

$$\lambda = \frac{1}{\xi} \left[\dot{\delta}^p \frac{\partial \hat{\xi}}{\partial \dot{\delta}^p} + \sum_{j=1}^{3} \left(\dot{\epsilon}^p_j \frac{\partial \hat{\xi}}{\partial \dot{\epsilon}^p_j} + \dot{\gamma}^p_j \frac{\partial \hat{\xi}}{\partial \dot{\gamma}^p_j} + n_j \frac{\partial \hat{\xi}}{\partial n_j}\right) + \frac{\partial \hat{\xi}}{\partial \mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i}} : \mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i}\right]. \tag{39}$$

Clearly, the evolution equations (Eqs. (38a)–(38c)) for the plastic strain attributes, anisotropy parameters, and the internal state variables are a set of implicit equations. In the above derivation, the rate of dissipation is maximized by keeping the stress attributes fixed while the strain rate

[4] Often termed as a microstress.

attributes and other kinematic variables are allowed to vary. It is possible to derive explicit evolution equations for the kinematic variables if the condition is reversed, *i.e.*, their conjugate kinetic variables are allowed to vary while the strain attributes and other kinematic variables are held fixed. This inversion is typically difficult for the plastic strain attributes and the associated stress unless a special form for the Helmholtz potential function ψ is assumed. Moreover, the evolution equations obtained thus far helps us to identify the thermodynamic conjugates corresponding to the anisotropy parameters n_j. If m_j denotes the microforce responsible for the change in anisotropy parameter $\overline{n}_j$, then in view of Eq. (38b), m_j can be defined as $m_j := \rho_0\, \partial\hat{\psi}/\partial\overline{n}_j$.

Because Eq. (20) is valid for the revised definition of strain attributes for anisotropic materials, it can be concluded that the difference between the total strain attributes and their corresponding plastic counterparts represents the elastic strain attributes or their linear combinations. Therefore, it is reasonable to assume that the Helmholtz potential function has the form

$$\begin{aligned}\psi = \hat{\psi}(l_{\mathcal{U}^e}, l_{\mathcal{U}^p}, \overline{n}_j) = \frac{1}{2}\ \Big[\ & N_{00}\,\overline{g}_{00}(\overline{n}_j)\,(\delta-\delta^p)^2 \\ &+ \sum_{i=1}^{3} N_{0i}\,\overline{g}_{oi}(\overline{n}_j)\,(\delta-\delta^p)\,(\varepsilon_i-\varepsilon_i^p) \\ &+ \sum_{i=1}^{3} N_{0(i+3)}\,\overline{g}_{0(i+3)}(\overline{n}_j)\,(\delta-\delta^p)\,(\gamma_i-\gamma_i^p) \\ &+ \sum_{\substack{i,j=1\\ i\le j}}^{3} N_{ij}\,\overline{g}_{ij}(\overline{n}_j)\,(\varepsilon_i-\varepsilon_i^p)\,(\varepsilon_j-\varepsilon_j^p) \\ &+ \sum_{i,j=1}^{3} N_{i(j+3)}\,\overline{g}_{i(j+3)}(\overline{n}_j)\,(\varepsilon_i-\varepsilon_i^p)\,(\gamma_j-\gamma_j^p) \\ &+ \sum_{\substack{i,j=1\\ i\le j}}^{3} N_{(i+3)(j+3)}\,(\gamma_i-\gamma_i^p)\,(\gamma_j-\gamma_j^p)\ \Big]\,.\end{aligned} \tag{40}$$

where the N's are material parameters and the $\overline{g}$'s are functions of the anisotropy parameters n_j. Here a decoupling of contributions from the anisotropy parameters and the strain attributes to the Helmholtz potential function is possible because the total and plastic strain attributes are related to components of the velocity gradient $\boldsymbol{\mathcal{L}}$ and the plastic velocity gradient $\boldsymbol{\mathcal{L}}^p$ through the same encoding/decoding maps,[5] respectively. The material parameters N are not all independent. This form for ψ leads to a Green elastic solid (*i.e.*, hyperelastic) response. With this assumed form for the Helmholtz potential function, the stress attributes can be written as

$$\begin{aligned}\pi &= \rho_0\frac{\partial\hat{\psi}}{\partial\delta} = -\rho_0\frac{\partial\hat{\psi}}{\partial\delta^p};\\ \sigma_j &= \rho_0\frac{\partial\hat{\psi}}{\partial\varepsilon_j} = -\rho_0\frac{\partial\hat{\psi}}{\partial\varepsilon_j^p};\\ \tau_j &= \rho_0\frac{\partial\hat{\psi}}{\partial\gamma_j} = -\rho_0\frac{\partial\hat{\psi}}{\partial\gamma_j^p}.\end{aligned} \tag{41}$$

Substituting these relations into the reduced rate of dissipation equation (36), we obtain

$$\pi\,\dot{\delta}^p + \sum_{j=1}^{3}\Big(\sigma_j\,\dot{\varepsilon}_j^p + \tau_j\,\dot{\gamma}_j^p - m_j\,n_j\Big) - \boldsymbol{\mathcal{P}}_i : \mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i} = \hat{\xi}. \tag{42}$$

[5] The encoding/decoding map that relates the total strain attributes to the velocity gradient $\boldsymbol{\mathcal{L}}$ involves the total stretch components a, b and c, whereas their plastic counterparts are used in the map between plastic strain attributes and the components of $\boldsymbol{\mathcal{L}}^p$. However, this difference does not deter us from decoupling the contribution of anisotropy parameters from that of the strain attributes, as the stretch components are used in expressing only the shear strain attributes in terms of their relevant components from the velocity gradient. These components, in turn, are free from the effects of an evolving anisotropy.

Because the reduced rate of dissipation criterion has now been expressed as a product of the kinematic and their conjugate kinetic attributes, it is now possible to carry out the maximization process with respect to either the set of kinematic variables, or their conjugate kinetic variables. Thereby, one can now derive a set of explicit evolution equations for the plastic strain attributes, anisotropy parameters, and the internal state variables.

Before deriving explicit evolution equations for the kinematic variables, we need to talk about the yield criterion of a material in this framework. Following the arguments of Srinivasa (2010) and Paul and Freed (2021), in view of the reduced rate of dissipation criterion (42), the yield criterion can be written as

$$Y(l_{km}) := \max_{l_{km}\neq\mathbf{0}} \frac{\pi\,\dot{\delta}^p + \sum_{j=1}^{3}\Big(\sigma_j\,\dot{\varepsilon}_j^p + \tau_j\,\dot{\gamma}_j^p - m_j\,n_j\Big) - \boldsymbol{\mathcal{P}}_i : \mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i}}{\overline{\xi}(l_S, l_{\mathcal{U}^p}, l_{\mathcal{V}^p})} = 1 \tag{43}$$

where l_{km} denotes the list of kinematic variables such that

$$l_{km} = \Big\{\ \dot{\delta}^p \quad \dot{\varepsilon}_1^p \quad \dot{\varepsilon}_2^p \quad \dot{\varepsilon}_3^p \quad \dot{\gamma}_1^p \quad \dot{\gamma}_2^p \quad \dot{\gamma}_3^p \quad n_1 \quad n_2 \quad n_3 \quad \mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i}\ \Big\}.$$

If the yield function $Y(l_{km}) < 1$ for some values of l_{km}, then the reduced rate of dissipation equation is violated, and therefore, the response of a material is elastic. One can easily show that this yield function $Y(l_{km})$ is convex in the l_{km} space.[6]

Let us now consider the case where the strain attributes, anisotropy parameters and the internal state variables are held fixed, while their corresponding kinetic variables are allowed to vary. Let us also define a list of variables l_{kt} that contains all the kinetic variables as

$$l_{kt} = \Big\{\ \pi \quad \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \tau_1 \quad \tau_2 \quad \tau_3 \quad m_1 \quad m_2 \quad m_3 \quad \boldsymbol{\mathcal{P}}_i\ \Big\}.$$

Note that here we consider the internal state variable to be of kinetic nature whereas its kinematic, thermodynamic conjugate is derived from employing the maximum rate of dissipation criterion. In this case, the yield function can be defined as

$$Y(l_{kt}) := \max_{l_{kt}\neq\mathbf{0}} \frac{\pi\,\dot{\delta}^p + \sum_{j=1}^{3}\Big(\sigma_j\,\dot{\varepsilon}_j^p + \tau_j\,\dot{\gamma}_j^p - m_j\,n_j\Big) - \boldsymbol{\mathcal{P}}_i : \mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i}}{\overline{\xi}(l_S, l_{\mathcal{U}^p}, l_{\mathcal{V}^p})} = 1. \tag{44}$$

One can also show that the yield function, defined in this way, is convex in the l_{kt} space. Now, a routine calculation to maximize the rate of dissipation function ξ with the reduced rate of dissipation constraint (42), taken with respect to the kinetic variables listed in l_{kt}, leads to

$$\begin{aligned}\dot{\delta}^p &= \mu\,\frac{\partial\xi}{\partial\pi}, \qquad \dot{\varepsilon}_i^p = \mu\,\frac{\partial\xi}{\partial\sigma_i}, \qquad \dot{\gamma}_i^p = \mu\,\frac{\partial\xi}{\partial\tau_i},\\ n_j &= \mu\,\frac{\partial\xi}{\partial m_j}, \qquad \mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i} = \mu\,\frac{\partial\xi}{\partial\boldsymbol{\mathcal{P}}_i}\end{aligned} \tag{45}$$

where μ is the consistency parameter that satisfies the condition that $\mu = 0$ whenever $Y(l_{kt}) < 1$. The consistency parameter μ can be determined by substituting the plastic strain-rate attributes $l_{\mathcal{U}^p}$, the anisotropy parameters n_j, and the co-rotational rate of internal state variable $\mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i}$ into the reduced rate of dissipation equation (42). Thus, Eq. (45) provides explicit expressions for the evolution of the kinematic variables. Geometrically, Eq. (45) implicates that the stress attributes lie along normals to the dissipation function at their corresponding strain rate attributes, whereas the microstresses m_j and $\boldsymbol{\mathcal{P}}_i$ that associate with the anisotropy parameters n_j and internal state variables $\boldsymbol{\mathcal{U}}^{\mathcal{A}_i}$ lie along normals to the dissipation function at their corresponding kinematic conjugates. For materials that exhibit a yielding behavior, the dissipation function is no longer differentiable at $l_{km} = \mathbf{0}$. Therefore, a Lagrange multiplier method cannot be used to maximize the rate of dissipation function for those materials. In that case, a standard method

[6] For a detailed derivation, see Paul and Freed (2021, § 3.5). The convexity for the additional variables can be established following a routine calculation as delineated in the mentioned article.

of convex analysis can be applied to show that the kinetic variables l_{kt} are along the normal cone to the convex hull $\mathbb{C}_{l_{km}}$ at $\hat{l}_{km}$, where $\hat{l}_{km}$ denotes a set of kinematic variables that satisfies the yield condition $Y(\hat{l}_{km}) = 1$ with the convex hull $\mathbb{C}_{l_{km}}$ being defined as

$$\mathbb{C}_{l_{km}} := \lambda_0\, \dot{\delta}^p + \sum_{j=1}^{3} \left(\lambda_j\, \dot{\varepsilon}_j^p + \lambda_{j+3}\, \dot{\gamma}_j^p + \lambda_{j+6}\, n_j \right) + \lambda_{10}\, \mathring{\boldsymbol{U}}^{\mathcal{A}_i}.$$

If l_{km} is another set of kinematic variables satisfying $Y(l_{km}) < 1$, then the flow rule can be written as

$$\pi(\dot{\hat{\delta}}^p - \dot{\delta}^p) + \sum_{j=1}^{3} \left[\sigma_j\, (\dot{\hat{\varepsilon}}_j - \dot{\varepsilon}_j) + \tau_j\, (\dot{\hat{\gamma}}_j - \dot{\gamma}_j) + m_j(\hat{n}_j - n_j) + \boldsymbol{\mathcal{P}} : (\mathring{\hat{\boldsymbol{U}}}^{\mathcal{A}_i} - \mathring{\boldsymbol{U}}^{\mathcal{A}_i}) \right] \geq 0. \tag{46}$$

Because it can be shown that the yield function is convex in the kinetic variable-space, it is also possible to obtain the normality rule in terms of the kinetic variables. It is also worth noting that the above derivation of the evolution equations holds for the internal state variables of other kinds (*i.e.*, scalars or vectors).

4.1. Examples

In this section, we demonstrate how internal state variables and a plastically-induced anisotropy enter into our framework by considering classical J_2 - plasticity. Specifically, we consider two problems. In the first problem, we examine the deformation of an elastic–plastic material that exhibits kinematic hardening. Here we further consider that the material behavior remains isotropic throughout the deformation process. In the second problem, we consider the plastic deformation of a material in which its anisotropy evolves with the deformation. Although in real-life problems, it is quite common to observe these two processes occurring simultaneously, here we have considered them separately for the sake of simplicity.

Kinematic hardening

Let us consider an elastic–plastic material that obeys a J_2 yield criterion and exhibits a Prager-like kinematic hardening (Prager, 1955). The material behavior is assumed to be isotropic throughout the deformation process, *i.e.*, $n_j = 1$. A Prager hardening rule assumes that the yield surface does not change in size and shape, but undergoes a translation in the direction of the strain increment. To keep track of the translation of this yield surface, we need to specify a backstress tensor that is incorporated in our model through a kinetic, internal, state variable. Let $\mathbf{b}$ denote a tensor-valued, internal, state variable that represents a backstress in the current configuration κ_t. One can transfer the backstress tensor into the configuration $\tilde{\kappa}_p$ using a suitable pull back operation and define its attributes corresponding to the seven modes of deformation in a similar fashion as in Eq. (14). Let β^π, β_i^σ and β_i^τ denote the backstress attributes corresponding to the dilatation, squeeze and shear modes of deformation, respectively. Let us further denote the rate of dissipation associated with the backstress tensor $\mathbf{b}$ by Ω. It is apparent that Ω is a function of the plastic strain rate attributes. Since J_2 plasticity considers a volume-preserving plastic deformation, the functional form of Ω can be written as

$$\Omega = \Omega(\dot{\varepsilon}_i^p, \dot{\gamma}_i^p), \quad i = 1, 2, 3. \tag{47}$$

A J_2 yield criterion is obtained through an appropriate choice for the rate of dissipation function ξ. Let us consider the rate of dissipation function as

$$\xi = k' \sqrt{\sum_{i=1}^{3} (\dot{\varepsilon}_i^{p^2} + \dot{\gamma}_i^{p^2})} + k'\, \Omega(\dot{\varepsilon}_i^p, \dot{\gamma}_i^p). \tag{48}$$

We further consider the special form for the Helmholtz potential function given in Eq. (40). Now, the stress attributes can be obtained from Eqs. (41) and (38a) as

$$\sigma_i = \frac{k \dot{\varepsilon}_i^p}{\sqrt{\sum_{i=1}^{3} (\dot{\varepsilon}_i^{p^2} + \dot{\gamma}_i^{p^2})}} + k\, \frac{\partial \Omega}{\partial \dot{\varepsilon}_i^p}; \quad \tau_i = \frac{k \dot{\gamma}_i^p}{\sqrt{\sum_{i=1}^{3} (\dot{\varepsilon}_i^{p^2} + \dot{\gamma}_i^{p^2})}} + k\, \frac{\partial \Omega}{\partial \dot{\gamma}_i^p} \tag{49}$$

where $k = \mu\, k'$ with μ being a consistency parameter. Since Ω is the rate of dissipation function associated with the backstress tensor, its derivatives with respect to the plastic strain rate attributes give us the corresponding backstress attributes, *i.e.*,

$$\beta_i^\sigma = k \frac{\partial \Omega}{\partial \dot{\varepsilon}_i^p}; \quad \beta_i^\tau = k \frac{\partial \Omega}{\partial \dot{\gamma}_i^p}. \tag{50}$$

Thus, Eq. (49) can be alternatively written as

$$\sigma_i = \frac{k \dot{\varepsilon}_i^p}{\sqrt{\sum_{i=1}^{3} (\dot{\varepsilon}_i^{p^2} + \dot{\gamma}_i^{p^2})}} + \beta_i^\sigma; \quad \tau_i = \frac{k \dot{\gamma}_i^p}{\sqrt{\sum_{i=1}^{3} (\dot{\varepsilon}_i^{p^2} + \dot{\gamma}_i^{p^2})}} + \beta_i^\tau. \tag{51}$$

From Eq. (51), the yield criterion can be written as

$$\sqrt{\sum_{i=1}^{3} \left[(\sigma_i - \beta_i^\sigma)^2 + (\tau_i - \beta_i^\tau)^2 \right]} = k \implies \frac{\sqrt{\sum_{i=1}^{3} \left[(\sigma_i - \beta_i^\sigma)^2 + (\tau_i - \beta_i^\tau)^2 \right]}}{k} = 1. \tag{52}$$

It is evident that Eq. (52) represents the J_2 yield criterion with Prager's kinematic hardening rule in our framework. The size and shape of the yield surface remains constant while it translates in the six-dimensional stress attributes space according to the evolution of the backstress attributes given in Eq. (50).

Note that in this example, we did not need to consider a co-rotational rate of the internal state variable as the internal variable is embedded in the same configuration in which constitutive equations are written. In fact, in our case, the plastic strain rate attributes act as the thermodynamic conjugates of the attributes of the internal variable (backstress). However. the same will not be true whenever a tensor-valued internal variable of kinematic nature is considered. In that case, the rate of dissipation function must be expressed in terms of the co-rotational rates of the internal state variables.

Plastically-induced anisotropy

Let us consider an example of a plastically-induced anisotropy within the J_2 plasticity framework. For the sake of simplicity, here we do not incorporate any internal state variable. In this case, the evolution of both the plastic squeeze and shear strain rate attributes and, the anisotropy parameters contributes to the rate of dissipation function. Therefore, the rate of dissipation function is chosen as

$$\xi = k' \sqrt{\sum_{i=1}^{3} (\dot{\varepsilon}_i^{p^2} + \dot{\gamma}_i^{p^2})} + t' \sqrt{\sum_{i=1}^{3} n_i^2}. \tag{53}$$

where $t' \sqrt{\sum_{i=1}^{3} n_i^2}$ is the additional rate of dissipation caused by evolving anisotropy. Recall that a coupling exists between the plastic squeeze strain rate attributes and the anisotropy parameters. Following the procedure in Appendix, the stress attributes can be obtained as

$$\sigma_i = \frac{k\, \dot{\varepsilon}_i^p}{\sqrt{\sum_{r=1}^{3} (\dot{\varepsilon}_r^{p^2} + \dot{\gamma}_r^{p^2})}} + \frac{t}{\sqrt{\sum_{r=1}^{3} n_r^2}} \cdot \sum_{j=1}^{3} n_j \frac{\partial n_j}{\partial \dot{\varepsilon}_i^p}; \tag{54a}$$

$$\tau_i = \frac{k\, \dot{\gamma}_i^p}{\sqrt{\sum_{r=1}^{3} (\dot{\varepsilon}_r^{p^2} + \dot{\gamma}_r^{p^2})}}; \tag{54b}$$

$$m_i = \frac{k}{\sqrt{\sum_{r=1}^{3} (\dot{\varepsilon}_r^{p^2} + \dot{\gamma}_r^{p^2})}} \sum_{j=1}^{3} \dot{\varepsilon}_j^p \frac{\partial \dot{\varepsilon}_j^p}{\partial n_i} + \frac{t\, n_i}{\sqrt{\sum_{j=1}^{3} n_j^2}} \tag{54c}$$

where $k = \mu\, k'$ and $t = \mu\, t'$ with μ being a consistency parameter. Since it is reasonable to assume that the functional dependence between the anisotropy parameters and the plastic squeeze strain rate attributes are known *a priori*, the corresponding stress attributes can be easily

determined. It is interesting to note that the squeeze stress attributes and the microstresses, m_i both contain terms associated with the two components of the assumed rate of dissipation function. This is a consequence of the coupling between anisotropy parameters and the plastic squeeze strain rate attributes. From Eq. (54a), one can write the yield criterion as

$$\sum_{i=1}^{3}\sqrt{(\sigma_i-\beta_i)^2+\tau_i^2}=k \implies \frac{\sum_{i=1}^{3}\sqrt{(\sigma_i-\beta_i)^2+\tau_i^2}}{k}=1 \tag{55}$$

where $\beta_i = \dfrac{t}{\sqrt{\sum_{r=1}^{3} n_r^2}}\sum_{j=1}^{3} n_j \dfrac{\partial n_j}{\partial \dot{\varepsilon}_i^p}$. The above yield criterion implies that for the assumed rate of dissipation function, the material undergoes a combined isotropic and kinematic hardening; a plastically-induced anisotropy results in a kinematic hardening whereas the evolution of the plastic squeeze and shear strain rate attributes leads to an isotropic hardening.

5. Summary

In this paper, we incorporated internal state variables that represent macroscopic manifestations of microstructural properties and plastically-induced anisotropy into our previously developed constitutive model. To develop this model, we use scalar conjugate stress/strain base pairs that arise from a **QR** decomposition of the deformation gradient. This upper-triangular decomposition results in an orthogonal rotation $\boldsymbol{\mathcal{R}}$ and an upper-triangular Laplace stretch $\boldsymbol{\mathcal{U}}$ that is further decomposed into elastic and plastic components. It has been shown that the intermediate configuration $\tilde{\kappa}_p$, which is related to the reference configuration of the body through the plastic Laplace stretch $\boldsymbol{\mathcal{U}}^p$, represents a macroscopic manifestation of the substructure of a constituent material. An appropriate co-rotational rate for a kinematic internal state variable $\boldsymbol{\mathcal{A}}_i$ (in the configuration $\tilde{\kappa}_p$) has been considered based on this intermediate configuration. Due to its importance in the context of plastically-induced anisotropy, here we have also considered the evolution of anisotropy during plastic deformation in our constitutive model. Traditionally, such evolution is considered through the internal state variable, whereas, in our case, this evolution is incorporated by considering the anisotropy parameters, used in an encoding/decoding map, as variables. Finally, a constitutive model for all the plastic strain attributes, anisotropy parameters, and internal state variables has been obtained by using a maximum rate of dissipation criterion.

CRediT authorship contribution statement

Sandipan Paul: Writing – review & editing, Writing – original draft, Visualization, Validation, Project administration, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix. Derivation of flow rules using a maximum rate of dissipation criterion

From Eq. (27), we observe that the anisotropy parameters are only associated with the dilatational and squeeze strain rates when they are expressed in terms of the components of the plastic velocity gradient $\boldsymbol{\mathcal{L}}^p$, or vice versa. Moreover, these strain rates are related to only three components of plastic Laplace stretch, $\mathcal{L}^p_{11}$, $\mathcal{L}^p_{22}$ and $\mathcal{L}^p_{33}$. Therefore, it is reasonable to carry out the maximization process with respect to these three components of the plastic Laplace stretch, the shear strain rates $\dot{\gamma}_j$, and the co-rotational rates of the internal state variables.

Now, from Eq. (37), the components of Laplace stretch in terms of the dilatational and squeeze strain rates and the anisotropy parameters can be written as

$$\mathcal{L}^p_{11} = n_1(\dot{\delta}^p + 2\dot{\varepsilon}^p_1 + \dot{\varepsilon}^p_2); \tag{A.1a}$$

$$\mathcal{L}^p_{22} = n_2(\dot{\delta}^p - \dot{\varepsilon}^p_1 + \dot{\varepsilon}^p_2); \tag{A.1b}$$

$$\mathcal{L}^p_{33} = n_3(\dot{\delta}^p - \dot{\varepsilon}^p_1 - 2\dot{\varepsilon}^p_2). \tag{A.1c}$$

The Lagrangian for our constrained optimization problem can therefore be written as

$$\mathbb{L} := \xi + \bar{\lambda}\left(\hat{\xi} + \rho_0\frac{\partial\hat{\psi}}{\partial\delta^p}\dot{\delta}^p + \rho_0\sum_{j=1}^{3}\left[\frac{\partial\hat{\psi}}{\partial\varepsilon^p_j}\dot{\varepsilon}^p_j + \rho_0\frac{\partial\hat{\psi}}{\partial\gamma_j}\dot{\gamma}_j + \frac{\partial\hat{\psi}}{\partial\bar{n}_j}n_j\right] + \boldsymbol{\mathcal{P}}_i : \mathring{\boldsymbol{\mathcal{U}}}^{\mathcal{A}_i}\right). \tag{A.2}$$

Now the condition for maximizing the Lagrangian $\mathbb{L}$ with respect to the component of plastic velocity gradient $\mathcal{L}^p_{11}$ is given as

$$\frac{\partial\mathbb{L}}{\partial\mathcal{L}^p_{11}} = 0 \implies (1+\bar{\lambda})\frac{\partial\xi}{\partial\mathcal{L}^p_{11}} + \rho_0\bar{\lambda}\left(\frac{\partial\hat{\psi}}{\partial\delta^p}\frac{\partial\dot{\delta}^p}{\partial\mathcal{L}^p_{11}} + \frac{\partial\hat{\psi}}{\partial\varepsilon^p_1}\frac{\partial\dot{\varepsilon}^p_1}{\partial\mathcal{L}^p_{11}} + \frac{\partial\hat{\psi}}{\partial\varepsilon^p_2}\frac{\partial\dot{\varepsilon}^p_2}{\partial\mathcal{L}^p_{11}} + \frac{\partial\hat{\psi}}{\partial\bar{n}_1}\frac{\partial n_1}{\partial\mathcal{L}^p_{11}}\right) = 0. \tag{A.3}$$

Note that the components $\mathcal{L}^p_{11}, \mathcal{L}^p_{22}$ and $\mathcal{L}^p_{33}$ do not explicitly depend upon the third squeeze strain rate $\dot{\varepsilon}^p_3$. This is due to the fact that the squeeze strain rate $\dot{\varepsilon}^p_3$ can be expressed as the linear combination of the other two. However, it does not pose any issue regarding the determination of an evolution equation for ε^p_3. Because $\mathcal{L}^p_{11}$ depends upon the strain rates and the anisotropy parameter n_1, Eq. (A.3) reduces to

$$\begin{aligned}(1+\bar{\lambda})\left[\frac{\partial\xi}{\partial\dot{\delta}^p}\frac{\partial\dot{\delta}^p}{\partial\mathcal{L}^p_{11}} + \frac{\partial\xi}{\partial\dot{\varepsilon}^p_1}\frac{\partial\dot{\varepsilon}^p_1}{\partial\mathcal{L}^p_{11}} + \frac{\partial\xi}{\partial\dot{\varepsilon}^p_2}\frac{\partial\dot{\varepsilon}^p_2}{\partial\mathcal{L}^p_{11}} + \frac{\partial\xi}{\partial n_1}\frac{\partial n_1}{\partial\mathcal{L}^p_{11}}\right] + \rho_0\bar{\lambda}\frac{\partial\hat{\psi}}{\partial\delta^p}\frac{\partial\dot{\delta}^p}{\partial\mathcal{L}^p_{11}} \\ + \rho_0\bar{\lambda}\frac{\partial\hat{\psi}}{\partial\varepsilon^p_1}\frac{\partial\dot{\varepsilon}^p_1}{\partial\mathcal{L}^p_{11}} + \rho_0\bar{\lambda}\frac{\partial\hat{\psi}}{\partial\varepsilon^p_2}\frac{\partial\dot{\varepsilon}^p_2}{\partial\mathcal{L}^p_{11}} + \rho_0\bar{\lambda}\frac{\partial\hat{\psi}}{\partial\bar{n}_1}\frac{\partial n_1}{\partial\mathcal{L}^p_{11}} = 0.\end{aligned} \tag{A.4}$$

Now substituting derivatives of the strain rates and the anisotropy parameter taken with respect to $\mathcal{L}^p_{11}$ into Eq. (A.4), we obtain

$$\begin{aligned}\frac{1}{n_1}\left[(1+\bar{\lambda})\frac{\partial\xi}{\partial\dot{\delta}^p} + \rho_0\bar{\lambda}\frac{\partial\psi}{\partial\delta^p}\right] + \frac{1}{2n_1}\left[(1+\bar{\lambda})\frac{\partial\xi}{\partial\dot{\varepsilon}^p_1} + \rho_0\bar{\lambda}\frac{\partial\psi}{\partial\varepsilon^p_1}\right] \\ + \frac{1}{n_1}\left[(1+\bar{\lambda})\frac{\partial\xi}{\partial\dot{\varepsilon}^p_2} + \rho_0\bar{\lambda}\frac{\partial\psi}{\partial\varepsilon^p_2}\right] + \frac{1}{\dot{\delta}^p + 2\dot{\varepsilon}^p_1 + \dot{\varepsilon}_2}\left[(1+\bar{\lambda})\frac{\partial\xi}{\partial n_1} + \rho_0\bar{\lambda}\frac{\partial\psi}{\partial\bar{n}_1}\right] = 0.\end{aligned} \tag{A.5}$$

Similarly, the condition for maximizing the Lagrangian $\mathbb{L}$ with respect to $\mathcal{L}^p_{22}$ and $\mathcal{L}^p_{33}$ are given as

$$\begin{aligned}\frac{1}{n_2}\left[(1+\bar{\lambda})\frac{\partial\xi}{\partial\dot{\delta}^p} + \rho_0\bar{\lambda}\frac{\partial\psi}{\partial\delta^p}\right] - \frac{1}{n_2}\left[(1+\bar{\lambda})\frac{\partial\xi}{\partial\dot{\varepsilon}^p_1} + \rho_0\bar{\lambda}\frac{\partial\psi}{\partial\varepsilon^p_1}\right] \\ + \frac{1}{n_2}\left[(1+\bar{\lambda})\frac{\partial\xi}{\partial\dot{\varepsilon}^p_2} + \rho_0\bar{\lambda}\frac{\partial\psi}{\partial\varepsilon^p_2}\right] + \frac{1}{\dot{\delta}^p - \dot{\varepsilon}^p_1 + \dot{\varepsilon}_2}\left[(1+\bar{\lambda})\frac{\partial\xi}{\partial n_1} + \rho_0\bar{\lambda}\frac{\partial\psi}{\partial\bar{n}_2}\right] = 0\end{aligned} \tag{A.6}$$

and

$$\begin{aligned}\frac{1}{n_3}\left[(1+\bar{\lambda})\frac{\partial\xi}{\partial\dot{\delta}^p} + \rho_0\bar{\lambda}\frac{\partial\psi}{\partial\delta^p}\right] - \frac{1}{n_3}\left[(1+\bar{\lambda})\frac{\partial\xi}{\partial\dot{\varepsilon}^p_1} + \rho_0\bar{\lambda}\frac{\partial\psi}{\partial\varepsilon^p_1}\right] \\ - \frac{1}{2n_3}\left[(1+\bar{\lambda})\frac{\partial\xi}{\partial\dot{\varepsilon}^p_2} + \rho_0\bar{\lambda}\frac{\partial\psi}{\partial\varepsilon^p_2}\right] + \frac{1}{\dot{\delta}^p - \dot{\varepsilon}^p_1 - 2\dot{\varepsilon}_2}\left[(1+\bar{\lambda})\frac{\partial\xi}{\partial n_1} + \rho_0\bar{\lambda}\frac{\partial\psi}{\partial\bar{n}_2}\right] = 0.\end{aligned}$$

(A.7)

The maximization process for the shear strain rates are rather straightforward. As they are not related to the components of $\mathcal{L}^p$ through the anisotropy parameters, one can carry out the maximization process directly with respect to the shear strain rates $\dot{\gamma}^p_j$. A routine calculation leads to

$$(1+\overline{\lambda})\frac{\partial \xi}{\partial \dot{\gamma}^p_j} = -\overline{\lambda}\,\rho_0\,\frac{\partial \psi}{\partial \gamma^p_j}. \tag{A.8}$$

Similarly, an evolution equation for the anisotropy parameters and the internal state variables are obtained as

$$(1+\overline{\lambda})\frac{\partial \xi}{\partial n_j} = -\overline{\lambda}\,\rho_0\,\frac{\partial \psi}{\partial \overline{n}_j} \tag{A.9}$$

and

$$(1+\overline{\lambda})\frac{\partial \xi}{\partial \overset{\circ}{\boldsymbol{U}}^{\mathcal{A}_i}} = -\overline{\lambda}\,\boldsymbol{\mathcal{P}}. \tag{A.10}$$

In view of Eq. (A.9), Eqs. (A.5), (A.6) and (A.7) can be collectively rewritten as

$$f_{1j}(n_j)\underbrace{\left[(1+\overline{\lambda})\frac{\partial \xi}{\partial \dot{\delta}^p}+\rho_0\,\overline{\lambda}\,\frac{\partial \psi}{\partial \delta^p}\right]}_{q_1(\delta^p,\dot{\delta}^p)}+f_{2j}(n_j)\underbrace{\left[(1+\overline{\lambda})\frac{\partial \xi}{\partial \dot{\epsilon}^p_1}+\rho_0\,\overline{\lambda}\,\frac{\partial \psi}{\partial \epsilon^p_1}\right]}_{q_2(\epsilon^p_1,\dot{\epsilon}^p_1)}$$
$$+\,f_{3j}(n_j)\underbrace{\left[(1+\overline{\lambda})\frac{\partial \xi}{\partial \dot{\epsilon}^p_2}+\rho_0\,\overline{\lambda}\,\frac{\partial \psi}{\partial \epsilon^p_2}\right]}_{q_3(\epsilon^p_2,\dot{\epsilon}^p_2)}=0 \tag{A.11}$$

where f_{ij}'s are functions of n_j in accordance with Eq. (37). Notice that for the reduced Eqs. (A.5), (A.6) and (A.7), their constituents q_i's remain the same. Moreover, it is evident that Eq. (A.11) must be satisfied for any variation of the dilatational and squeeze strain attributes and their rates in order to maximize the Lagrangian $\mathbb{L}$. Moreover, in each of the constituents of Eq. (A.11), q_i depends on a certain mode of deformation, for example, the constituent q_1 depends only on the dilatational mode of deformation. Because the dilatation and these squeeze modes of deformation are independent of each other, one can vary the functions q_1, q_2 and q_3 arbitrarily such that Eq. (A.11) is always satisfied. This is possible if and only if these constituents are individually zero, *i.e.*, $q_1(\delta^p,\dot{\delta}^p) = q_2(\epsilon^p_1,\dot{\epsilon}^p_1) = q_3(\epsilon^p_3,\dot{\epsilon}^p_3) = 0$. Thus, the condition to maximize the Lagrangian $\mathbb{L}$ with respect to $\mathcal{L}^p_{11}$, $\mathcal{L}^p_{22}$ and $\mathcal{L}^p_{33}$ can be written as

$$\begin{aligned}
(1+\overline{\lambda})\frac{\partial \xi}{\partial \dot{\delta}^p} &= -\rho_0\,\overline{\lambda}\,\frac{\partial \psi}{\partial \delta^p};\\
(1+\overline{\lambda})\frac{\partial \xi}{\partial \dot{\epsilon}^p_1} &= -\rho_0\,\overline{\lambda}\,\frac{\partial \psi}{\partial \epsilon^p_1};\\
(1+\overline{\lambda})\frac{\partial \xi}{\partial \dot{\epsilon}^p_2} &= -\rho_0\,\overline{\lambda}\,\frac{\partial \psi}{\partial \epsilon^p_2}.
\end{aligned} \tag{A.12}$$

Because the third squeeze mode satisfies the condition $\epsilon^p_3 = -(\epsilon^p_1+\epsilon^p_2)$ and $\dot{\epsilon}^p_3 = -(\dot{\epsilon}^p_1+\dot{\epsilon}^p_2)$, the evolution equation for ϵ^p_3 can be written as

$$(1+\overline{\lambda})\frac{\partial \xi}{\partial \dot{\epsilon}^p_3} = -\rho_0\,\overline{\lambda}\,\frac{\partial \psi}{\partial \epsilon^p_3}. \tag{A.13}$$

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